

THE FUTURE ASYMPTOTICS OF BIANCHI VIII VACUUM SOLUTIONS

HANS RINGSTRÖM

ABSTRACT. Bianchi VIII vacuum solutions to Einstein's equations are causally geodesically complete to the future, given an appropriate time orientation, and the objective of this article is to analyze the asymptotic behaviour of solutions in this time direction. For the Bianchi class A spacetimes, there is a formulation of the field equations that was presented in an article by Wainwright and Hsu, and we will analyze the asymptotic behaviour of solutions in these variables. We also try to give the analytic results a geometric interpretation by analyzing how a normalized version of the Riemannian metric on the spatial hypersurfaces of homogeneity evolves.

1. INTRODUCTION

Let us begin by defining what we mean by Bianchi class A spacetimes. We do this by defining the concept class A vacuum initial data. Let us introduce some terminology. Let G be a 3-dimensional Lie group, e_i , $i = 1, 2, 3$ be a basis of the Lie algebra with structure constants determined by $[e_i, e_j] = \gamma_{ij}^k e_k$. If $\gamma_{ik}^k = 0$, then the Lie algebra and Lie group are said to be of class A and

$$(1) \quad \gamma_{ij}^k = \epsilon_{ijm} n^{km}$$

where the symmetric matrix n^{ij} is given by

$$(2) \quad n^{ij} = \frac{1}{2} \gamma_{kl}^{(i} \epsilon^{j)kl}.$$

Definition 1. *Class A vacuum initial data* for Einstein's equations consist of the following. A Lie group G of class A, a left invariant metric g on G and a left invariant symmetric covariant two-tensor k on G satisfying

$$(3) \quad R_g - k_{ij} k^{ij} + (\text{tr}_g k)^2 = 0$$

and

$$(4) \quad \nabla_i \text{tr}_g k - \nabla^j k_{ij} = 0$$

where ∇ is the Levi-Civita connection of g and R_g is the corresponding scalar curvature, indices are raised and lowered by g .

Consider class A vacuum initial data. We can choose a left invariant orthonormal basis $\{e_i\}$ with respect to g so that the corresponding matrix n^{ij} defined in (2) is diagonal with diagonal elements n_1 , n_2 and n_3 . By an appropriate choice of orthonormal basis n_1, n_2, n_3 can be assumed to belong to one and only one of the types given in table 1. We assign a Bianchi type to the initial data accordingly. This division constitutes a classification of the class A Lie algebras, see Lemma 3.

TABLE 1. Bianchi class A.

Type	n_1	n_2	n_3
I	0	0	0
II	+	0	0
VI ₀	0	+	−
VII ₀	0	+	+
VIII	−	+	+
IX	+	+	+

We can divide the initial data into three classes,

1. Bianchi IX initial data. In this case the maximal globally hyperbolic development of the initial data is future and past causally geodesically incomplete. It expands for a certain period of time, reaches a moment of maximal expansion and then recollapses.
2. Bianchi I and VII₀ initial data with $\text{tr}_g k = 0$. The corresponding developments are causally geodesically complete.
3. The remaining types of initial data. With a suitable time orientation such data evolve into a future causally geodesically complete spacetime expanding forever into the future. The development is however past causally geodesically incomplete.

In this paper we are interested in the third class of initial data.

There is a formulation of Einstein's equations due to Ellis and MacCallum [1] covering, among other things, the Bianchi class A spacetimes. In an article by Wainwright and Hsu [2], a normalized version of the corresponding variables together with a different time coordinate were introduced. The main part of this paper consists of an analysis of the asymptotic behaviour of these variables. Our main interest is in Bianchi VIII, but we also consider the other Bianchi class A spacetimes in the third class as above. The Bianchi I, II and VI₀ cases are quickly handled, and are included for completeness, but it takes some time to analyze the asymptotics of Bianchi VII₀. Observe that Bianchi VII₀ perfect fluids have been considered in [3]. Bianchi VIII is dealt with definitively, in the sense that it is proven to what the variables of Wainwright and Hsu converge. However, in certain situations this may not necessarily be enough. For instance, it is proven that $N_2 + N_3$ converge to infinity, if $N_2, N_3 > 0$ and $N_1 < 0$, and that $N_2 - N_3$ converge to zero. It is conceivable that if one is interested in computing curvature invariants for instance, one has to consider expressions of the form $N_2^2 - N_3^2$, concerning which the results contained in this paper say nothing.

Let us also mention that even though it does not require much time to sort out the asymptotics for Bianchi VI₀, the results yield an interesting consequence. In fact, one gets the conclusion that Bianchi VIII is very different from Bianchi IX as far as the behaviour towards the singularity is concerned. In [4] it was proven that generic Bianchi IX solutions converge to an attractor, where by the attractor, we mean the set of vacuum type I and II points, and the generic points are obtained by subtracting a finite number of positive codimension submanifolds from the manifold of Bianchi type IX points. It was also proven that for generic Bianchi IX solutions, the convergence to the attractor is almost monotone in the following sense. Given

an $\epsilon > 0$, there is a $\delta > 0$ such that if x constitutes Bianchi IX initial data closer to the attractor than δ , then applying the flow to x in the past time direction will not result in points further away from the attractor than ϵ . By distance, we here mean the ordinary Euclidean distance in the Wainwright Hsu variables in \mathbb{R}^6 , the relevant space in the Bianchi IX matter case. This result, together with the result that generic Bianchi IX solutions have an α -limit point on the Kasner circle, yield the conclusion that generic solutions converge to the attractor. The almost monotone convergence constitutes the most difficult step. In Proposition 3, we prove that the corresponding statement for Bianchi VIII is not true. In fact, we prove that there is a fixed number $\eta > 0$, such that for any $\epsilon > 0$ we can find Bianchi VIII initial data closer to the Kasner circle than ϵ which will at one point to the past be at a distance further than η away from the attractor. This does of course not prove that Bianchi VIII solutions may not converge to the attractor, but it does imply that the main part of the argument concerning Bianchi IX presented in [4] is of no help in proving convergence to the attractor, and that a completely different method is required, assuming one still believes that generic Bianchi VIII solutions converge to the attractor towards the singularity.

Let us try to explain the argument concerning the future asymptotics of a Bianchi VIII solution with $N_1 < 0$ and $N_2, N_3 > 0$. By a monotonicity argument, one can prove that $N_1 \rightarrow 0$ and $N_2, N_3 \rightarrow \infty$ as $\tau \rightarrow \infty$. This is a very important observation in that it introduces a natural concept of order of magnitude; powers of $N_2 + N_3$. Writing the constraint as

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + (N_2 - N_3)^2 - 2N_1(N_2 + N_3)] = 1,$$

we conclude that all the terms involved are bounded since they are all non-negative. The derivatives of Σ_+ and $N_1(N_2 + N_3)$ are bounded by constants independent of the initial data, but we have

$$\tilde{x}' = -3(N_2 + N_3)\tilde{y} + \dots, \quad \tilde{y}' = 3(N_2 + N_3)\tilde{x} + \dots$$

where $\tilde{x} = \Sigma_-$ and $\tilde{y} = \sqrt{3}(N_2 - N_3)/2$, and the dots represent expressions that can be bounded by constants independent of which Bianchi VIII solution we are considering. Thus, if $(\tilde{x}^2 + \tilde{y}^2)^{1/2}$ is big in comparison with $(N_2 + N_3)^{-1}$, \tilde{y} and \tilde{x} will essentially behave as sine and cosine, and the frequency of the oscillation will go to infinity as $\tau \rightarrow \infty$. The oscillations in \tilde{x} and \tilde{y} are not so interesting, but the evolution in Σ_+ and $N_1(N_2 + N_3)$ is. For that reason, it is natural to consider what happens from, say, one time at which $\Sigma_- = 0$ to the next. Before doing so, some effort has to be put into estimating the error committed in approximating \tilde{y} and \tilde{x} with sine and cosine, into proving that there actually are zeros of Σ_- and estimating the variation of different objects during the time that elapses between the zeros. After having done that, it will be possible to express Σ_+ and $N_1(N_2 + N_3)$ at a zero of Σ_- in terms of the values of the same expressions in the previous zero. In other words, it will be possible to replace the continuous flow with a discrete map depending on only two variables. This approximation will not always be valid, but when it is not, it can be replaced by less sophisticated arguments. The discrete map is then the main tool in a sequence of technical arguments proving the desired result.

When we evolve initial data we get a Lorentz metric $\bar{g} = -dt^2 + g(t)$ on $I \times G$ where I is an open interval and $g(t)$ is a Riemannian left invariant metric on G . Let Γ be a

subgroup of the group of diffeomorphisms of G , acting properly discontinuously on G . We can consider Γ as acting on $I \times G$ by letting Γ act only on G . In situations where $g(t)$ is invariant under Γ , we can take the quotient to get a solution to Einstein's equations on $I \times M$, where $M = G/\Gamma$. Assuming M to be compact we can define the reduced Hamiltonian, c.f. [5],

$$H_{\text{reduced}} = -(\text{tr}_g k)^3 \text{vol}(M, g),$$

where $k(t)$ is the second fundamental form of $\{t\} \times M$. We can consider it to be a function of t . This object is of interest in the context of the work by Fischer and Moncrief [5]. We prove that the reduced Hamiltonian converges to zero for all the initial data in class three, though it should be remarked that the spatial hypersurfaces should be of Yamabe type -1, see [5], for this result to be of interest. Finally, let us refer the reader to [6] and references therein for further discussion of Bianchi VIII vacuum solutions.

The equations of Wainwright and Hsu are to be found in section 2. We mention some properties and describe an important tool in the analysis of the asymptotics of solutions to these equations; the monotonicity principle. In section 3 we relate the Lorentz metric on the development with the Wainwright Hsu variables. A formula expressing the reduced Hamiltonian in terms of these variables is also given. The main part of this paper consists of an analysis of the asymptotic behaviour of solutions to the equations of Wainwright and Hsu. This analysis occupies the remaining sections except for the last. The final section contains an analysis of the asymptotic behaviour of the Riemannian metric on the spatial hypersurfaces of homogeneity.

2. THE EQUATIONS OF WAINWRIGHT AND HSU

We formulate the equations we will use here and state some properties. The following section contains a derivation. Einstein's vacuum equations take the following form in the formulation due to Wainwright and Hsu

$$\begin{aligned} N'_1 &= (q - 4\Sigma_+)N_1 \\ N'_2 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \\ N'_3 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3 \\ \Sigma'_+ &= -(2 - q)\Sigma_+ - 3S_+ \\ \Sigma'_- &= -(2 - q)\Sigma_- - 3S_- \end{aligned} \tag{5}$$

where the prime denotes derivative with respect to τ and

$$\begin{aligned} q &= 2(\Sigma_+^2 + \Sigma_-^2) \\ S_+ &= \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)] \\ S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3). \end{aligned} \tag{6}$$

The vacuum Hamiltonian constraint is

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_1N_3)] = 1. \tag{7}$$

The above equations have certain symmetries described in Wainwright and Hsu [2]. By permuting N_1, N_2, N_3 arbitrarily we get new solutions if we at the same time

carry out appropriate combinations of rotations by integer multiples of $2\pi/3$ and reflections in the (Σ_+, Σ_-) -plane. Changing the sign of all the N_i at the same time does not change the equations. Classify points $(N_1, N_2, N_3, \Sigma_+, \Sigma_-)$ according to the values of N_1, N_2, N_3 in the same way as in table 1. Since the sets $N_i > 0$, $N_i < 0$ and $N_i = 0$ are invariant under the flow of the equation we may classify solutions to (5)-(7) accordingly. When we speak of Bianchi VIII solutions we will assume two $N_i > 0$ and one < 0 . The *Kasner circle* is the subset of $\Sigma_+ \Sigma_- N_1 N_2 N_3$ -space where $N_i = 0$ and $\Sigma_+^2 + \Sigma_-^2 = 1$.

We only consider solutions to (5)-(7) which are not of Bianchi IX type. By the constraint (7) we conclude that $q \leq 2$ for the entire solution. As a consequence the N_i cannot grow faster than exponentially due to (5). The solutions we consider can thus not blow up in a finite time so that we have existence intervals of the form $(-\infty, \infty)$.

Definition 2. Let \mathcal{K}_1 , \mathcal{K}_2 and \mathcal{K}_3 be defined as the subset of the Kasner circle on which $q - 4\Sigma_+ < 0$, $q + 2\Sigma_+ + 2\sqrt{3}\Sigma_- < 0$ and $q + 2\Sigma_+ - 2\sqrt{3}\Sigma_- < 0$ respectively, c.f. (5).

The set $\Sigma_- = 0$, $N_2 = N_3$ is invariant under the flow of (5)-(7). Applying the symmetries to this set we get new invariant sets.

The concepts α - and ω -limit set will be useful.

Definition 3. Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and consider a solution x to the equation

$$(8) \quad \frac{dx}{dt} = f \circ x, \quad x(0) = x_0,$$

with maximal existence interval (t_-, t_+) . We call a point x_* an ω -*limit point* of the solution x , if there is a sequence $t_k \rightarrow t_+$ with $x(t_k) \rightarrow x_*$. The ω -*limit set* of x is the set of its ω -limit points. The α -limit set is defined similarly by replacing t_+ with t_- .

Lemma 1. *Let f and x be as in Definition 3. The ω -limit set of x is closed and invariant under the flow of f . If there is a T such that $x(t)$ is contained in a compact set for $t \geq T$, then the ω -limit set of x is connected.*

Proof. See e.g. [7]. \square

The following lemma will be a basic tool in the analysis of the asymptotics, we will refer to it as *the monotonicity principle*.

Lemma 2. *Consider (8). Let U be an open subset of \mathbb{R}^n and M a closed subset which is invariant under the flow of the vectorfield f . Assume there is a function $F \in C(U, \mathbb{R})$ such that $F(x(t))$ is strictly monotone for any solution $x(t)$ of (8), as long as $x(t) \in U \cap M$. Then no solution of (8) whose image is contained in $U \cap M$ has an α - or ω -limit point in U .*

Remark. Observe that one can use $M = \mathbb{R}^n$. We will mainly choose M to be the closed invariant submanifold of \mathbb{R}^5 defined by (7). If one N_i is zero and two are non-zero we consider the number of variables to be four etc.

Proof. Let x be a solution to (8) which is contained in $U \cap M$, and which has maximal existence interval (t_-, t_+) . Then $F \circ x$ is strictly monotone. Suppose $p \in U$ is an

ω -limit point of x , so that there is a sequence $t_n \rightarrow t_+$ such that $x(t_n) \rightarrow p$. Thus $F(x(t_n)) \rightarrow F(p)$, but $F \circ x$ is monotone so that $F(x(t)) \rightarrow F(p)$ as $t \rightarrow t_+$. Thus $F(q) = F(p)$ for all ω -limit points q of x . Since M is closed $p \in M$. The solution \bar{x} of (8) with initial value p is contained in M by the invariance property of M , it consists of ω -limit points of x so that $F(\bar{x}(t)) = F(p)$ which is constant. Furthermore, on an open set containing zero it takes values in U contradicting the assumptions of the lemma. The argument for the α -limit set is similar. \square

3. RELATION BETWEEN THE VARIABLES OF WAINWRIGHT AND HSU AND THE SPACE TIME METRIC

We need to relate the evolution of the variables in the formulation due to Wainwright and Hsu to the evolution of the corresponding spacetime. In order to do this we first derive the formulation first presented by Ellis and MacCallum [1]. Then we derive equations (5)-(7) as in [2]. Secondly, we turn the argument around and start with solutions to the equations of Ellis and MacCallum and then construct the spacetime metric. The second part uses the first part. We can then express the spacetime metric in terms of the variables of Wainwright and Hsu. Finally, we find expressions for the reduced Hamiltonian in terms of the Wainwright Hsu variables.

3.1. Derivation of the equations of Wainwright and Hsu. Consider first a special class of spatially homogeneous four dimensional Lorentz manifolds of the form

$$(9) \quad (I \times G, -dt^2 + \chi_{ij}(t)\xi^i \otimes \xi^j)$$

where I is an open interval, G is a Lie group of class A, χ_{ij} is a smooth positive definite matrix and the ξ^i are the duals of a left invariant basis on G .

We express the condition that a manifold of the form (9) satisfies Einstein's vacuum equations in terms of the variables of Ellis and MacCallum. As these variables are defined in terms of a suitable orthonormal basis, we begin by constructing it. Let $e_0 = \partial_t$ and $e_i = a_i^j Z_j$, $i=1,2,3$, be an orthonormal basis, where a is a C^∞ matrix valued function of t and the Z_i are the duals of ξ^i . Below, Latin indices will be raised and lowered by δ_{ij} .

By the following argument we can assume $\langle \nabla_{e_0} e_i, e_j \rangle = 0$. Let the matrix A satisfy $e_0(A) + AB = 0$, $A(0) = \text{Id}$ where $B_{ij} = \langle \nabla_{e_0} e_i, e_j \rangle$ and Id is the 3×3 identity matrix. Then A is smooth and $SO(3)$ valued and if $e'_i = A_i^j e_j$ then $\langle \nabla_{e_0} e'_i, e'_j \rangle = 0$.

Let

$$(10) \quad \theta(X, Y) = \langle \nabla_X e_0, Y \rangle,$$

$\theta_{\alpha\beta} = \theta(e_\alpha, e_\beta)$ and $[e_\beta, e_\delta] = \gamma_{\beta\delta}^\alpha e_\alpha$ where Greek indices run from 0 to 3. The objects $\theta_{\alpha\beta}$ and $\gamma_{\beta\delta}^\alpha$ will be viewed as smooth functions from I to some suitable \mathbb{R}^k and the variables will be defined in terms of them.

Observe that $[Z_i, e_0] = 0$. The e_i span the tangent space of G and $\langle [e_0, e_i], e_0 \rangle = 0$. We get $\theta_{00} = \theta_{0i} = 0$ and $\theta_{\alpha\beta}$ symmetric. We also have $\gamma_{ij}^0 = \gamma_{0i}^0 = 0$ and

$\gamma_{0j}^i = -\theta_{ij}$. We let n be defined as in (2) and

$$\sigma_{ij} = \theta_{ij} - \frac{1}{3}\theta\delta_{ij}$$

where we by abuse of notation have written $\text{tr}(\theta)$ as θ .

We compute the Einstein tensor in terms of n , σ and θ . The Jacobi identities for e_α yield

$$(11) \quad e_0(n_{ij}) - 2n_{k(i}\sigma_{j)}^k + \frac{1}{3}\theta n_{ij} = 0.$$

The $0i$ -components of the Einstein equations are equivalent to

$$(12) \quad \sigma_i^k n_{kj} - n_i^k \sigma_{kj} = 0.$$

Letting $b_{ij} = 2n_i^k n_{kj} - \text{tr}(n)n_{ij}$ and $s_{ij} = b_{ij} - \frac{1}{3}\text{tr}(b)\delta_{ij}$ the trace free part of the ij equations are

$$(13) \quad e_0(\sigma_{ij}) + \theta\sigma_{ij} + s_{ij} = 0.$$

The fact that $R_{00} = 0$ yields the Raychaudhuri equation

$$(14) \quad e_0(\theta) + \theta_{ij}\theta^{ij} = 0$$

and using this together with the trace of the ij -equations yields a constraint

$$(15) \quad \sigma_{ij}\sigma^{ij} + (n_{ij}n^{ij} - \frac{1}{2}\text{tr}(n)^2) = \frac{2}{3}\theta^2.$$

Equations (11)-(15) are special cases of equations given in Ellis and MacCallum [1]. At $t = 0$ we may diagonalize n and σ simultaneously since they commute (12). Rotating e_α by the corresponding element of $SO(3)$ yields upon going through the definitions that the new n and σ are diagonal at $t = 0$. Collect the off-diagonal terms of n and σ in one vector v . By (11) and (13) there is a time dependent matrix C such that $\dot{v} = Cv$ so that $v(t) = 0 \forall t$ since $v(0) = 0$. Since the rotation was time independent $\langle \nabla_{e_0} e_i, e_j \rangle = 0$ holds in the new basis.

Introduce, as in Wainwright and Hsu [2], $\Sigma_{ij} = \sigma_{ij}/\theta$, $N_{ij} = n_{ij}/\theta$, $B_{ij} = 2N_i^k N_{kj} - N_k^k N_{ij}$ and $S_{ij} = B_{ij} - \frac{1}{3}B_k^k \delta_{ij}$ and define a new time coordinate τ , independent of time orientation, by

$$(16) \quad \frac{dt}{d\tau} = \frac{3}{\theta}.$$

Let $\Sigma_+ = \frac{3}{2}(\Sigma_{22} + \Sigma_{33})$ and $\Sigma_- = \sqrt{3}(\Sigma_{22} - \Sigma_{33})/2$. If we let N_i be the diagonal elements of N_{ij} , equations (11) and (13) turn into (5)-(7) The Raychaudhuri equation (14) takes the form

$$(17) \quad \theta' = -(1 + q)\theta.$$

3.2. Construction of the spacetime. Above we derived Einstein's equations assuming the spacetime to be of a special form. Next we construct a spacetime of the form (9) beginning with initial data as in Definition 1. The construction allows us to express the evolution of the metric in terms of the variables of Wainwright and Hsu. One step is the following lemma, which also classifies the class A Lie algebras.

Lemma 3. *Table 1 constitutes a classification of the class A Lie algebras. Consider an arbitrary basis $\{e_i\}$ of the Lie algebra. Then by applying an orthogonal matrix to it, we can construct a basis $\{e'_i\}$ such that the corresponding n' defined by (2) has diagonal elements of one of the types given in table 1.*

Proof. Let e_i be a basis for the Lie algebra and n be defined as in (2). If we change the basis according to $e'_i = (A^{-1})^j_i e_j$ then n transforms to

$$(18) \quad n' = (\det A)^{-1} A^t n A$$

Since n is symmetric we assume from here on that the basis is such that it is diagonal. The matrix $A = \text{diag}(1 \ 1 \ -1)$ changes the sign of n . A suitable orthogonal matrix performs even permutations of the diagonal. The number of non-zero elements on the diagonal is invariant under transformations (18) taking one diagonal matrix to another. If $A = (a_{ij})$ and the diagonal matrix n' is constructed as in (18) we have $n'_{kk} = (\det A)^{-1} \sum_{i=1}^3 a_{ik}^2 n_{ii}$ so that if all the diagonal elements of n have the same sign the same is true for n' . The statements of the lemma follow. \square

Consider class A vacuum initial data (G, g, k) with notation as in Definition 1. We construct a spacetime of the form (9) whose induced metric and second fundamental form on $\{0\} \times G$ is g and k . Let e'_i , $i = 1, 2, 3$ be a left invariant orthonormal basis. We can assume the corresponding n' to be of one of the forms given in table 1 by Lemma 3. The content of (4) is that $k_{ij} = k(e'_i, e'_j)$ and n' are to commute. We may thus also assume k_{ij} to be diagonal without changing the earlier conditions of the construction. If we let $n(0) = n'$, $\theta(0) = -\text{tr}_g k$ and $\sigma_{ij}(0) = -k_{ij} + \theta \delta_{ij}/3$ then (3) is the same as (15). Let n , σ and θ satisfy (11), (13) and (14) with initial values as specified above. Since (15) is satisfied at $t = 0$ it is satisfied for all times. For reasons given in connection with (15) n and σ will remain diagonal so that (12) will always hold.

Let $M = I \times G$ where I is the maximal existence interval for solutions to (11)-(15) with initial data as above. We construct a basis e_α with the same properties as the basis used in deriving (11)-(15). Then we define a metric by demanding that the basis be orthonormal and show that the corresponding \tilde{n} , $\tilde{\sigma}$ and $\tilde{\theta}$ coincide with n , σ and θ . We will thereby have constructed a Lorentz manifold satisfying Einstein's vacuum equations with the correct initial conditions.

Let n_i and σ_i denote the diagonal elements of n and σ respectively. Let $f_i(0) = 1$ and $\dot{f}_i/f_i = 2\sigma_i - \theta/3$. Let $a_i = (\prod_{j \neq i} f_j)^{1/2}$ and define $e_i = a_i e'_i$. Then \tilde{n} associated to e_i equals n . We complete the basis by letting $e_0 = \partial_t$. Define a metric $\langle \cdot, \cdot \rangle$ on M by demanding e_α to be orthonormal with $\langle e_0, e_0 \rangle = -1$ and $\langle e_i, e_i \rangle = 1$, $i = 1, 2, 3$ and let ∇ be the associated Levi-Civita connection.

Compute $\langle \nabla_{e_0} e_i, e_j \rangle = 0$. If $\tilde{\theta}(X, Y) = \langle \nabla_X e_0, Y \rangle$ and $\tilde{\theta}_{\mu\nu} = \tilde{\theta}(e_\mu, e_\nu)$, then $\tilde{\theta}_{00} = \tilde{\theta}_{i0} = \tilde{\theta}_{0i} = 0$. Furthermore,

$$\frac{1}{a_j} e_0(a_j) \delta_{ij} = -\tilde{\theta}_{ij}$$

(no summation over j) so that $\tilde{\theta}_{ij}$ is diagonal and $\text{tr} \tilde{\theta} = \theta$. Finally,

$$-\tilde{\sigma}_{ii} = -\tilde{\theta}_{ii} + \frac{1}{3} \theta = -\sigma_i.$$

The constructed Lorentz manifold thus satisfies Einstein's vacuum equations. By the above we have.

Lemma 4. *Given initial data as in Definition 1, we get a solution to Einstein's equations*

$$(19) \quad \bar{g} = -dt^2 + \sum_{i=1}^3 a_i^{-2}(t) \xi^i \otimes \xi^i$$

where ξ^i are the duals of a basis of the Lie algebra of G and

$$(20) \quad a_i(t) = \exp\left(-\int_0^t (\sigma_i + \frac{1}{3}\theta) ds\right)$$

where θ, σ_i (and n_i) constitute solutions to (11)-(15) with suitable initial conditions. If I is the corresponding existence interval, the metric (19) is defined on the manifold $I \times G$.

By [4] this metric is globally hyperbolic, future causally geodesically complete and $\tau \rightarrow \infty$ corresponds to the geodesically complete direction, assuming the initial data are not of type IX.

3.3. The rescaled metric. The equations (19) and (20) express the spacetime metric in terms of the Ellis-MacCallum variables. Let $g(t)$ denote the Riemannian metric induced on $\{t\} \times G$, which we will also consider to be a metric on G . Define a rescaled version of this metric by

$$(21) \quad \tilde{g}(t) = (a_1 a_2 a_3)^{2/3}(t) \sum_{i=1}^3 a_i^{-2}(t) \xi^i \otimes \xi^i.$$

We can write it as

$$(22) \quad \tilde{g}(t) = \sum_{i=1}^3 \lambda_i(t) \xi^i \otimes \xi^i$$

where

$$\lambda_i(t) = \exp\left(\int_0^t 2\sigma_i ds\right).$$

Defining a new time coordinate by

$$(23) \quad \frac{dt}{d\tau} = \frac{3}{\theta}, \quad \tau(0) = 0$$

we get

$$\int_0^t 2\sigma_i ds = \int_0^\tau 2\sigma_i \frac{3}{\theta} d\tau' = \int_0^\tau 6\Sigma_i d\tau'.$$

Thus if we consider the λ_i to be functions of τ we have

$$(24) \quad \lambda_i(\tau) = \exp\left(\int_0^\tau 6\Sigma_i d\tau'\right).$$

Similarly,

$$(25) \quad a_i(\tau) = \exp\left(-\int_0^\tau (3\Sigma_i + 1) ds\right).$$

However, the Σ_i can be expressed in terms of Σ_+ and Σ_- . We have

$$(26) \quad \Sigma_1 = -\frac{2}{3}\Sigma_+, \quad \Sigma_2 = \frac{1}{3}\Sigma_+ + \frac{1}{\sqrt{3}}\Sigma_-, \quad \Sigma_3 = \frac{1}{3}\Sigma_+ - \frac{1}{\sqrt{3}}\Sigma_-.$$

Thus, if we know what Σ_+ and Σ_- converge to as $\tau \rightarrow \infty$, we roughly know how \tilde{g} behaves asymptotically.

3.4. The reduced Hamiltonian. As mentioned in the introduction, there is in some situations a subgroup Γ of the group of diffeomorphisms of G , acting properly discontinuously on G , such that $g(t)$ is invariant under Γ for all t . In such situations, we can consider g to be a metric on $M = G/\Gamma$. Observe however that the basis e'_i need not necessarily descend to a basis on M . Let us assume M to be compact. We can then consider the reduced Hamiltonian

$$H(t) = -[-\theta(t)]^3 V(t)$$

where $V(t) = \text{vol}(M, g(t))$.

Lemma 5. *In terms of the time coordinate τ defined by (23) we have*

$$(27) \quad H(\tau) = \theta^3(0)V(0) \exp(-3 \int_0^\tau q ds).$$

Proof. Since M is compact we can choose a finite partition of unity ϕ_i $i = 1, \dots, k$ subordinated to coordinate charts. We can then write

$$V(t) = \sum_{i=1}^k \int_M \phi_i \sqrt{\det g(t)(\partial_j, \partial_m)} dx^1 \dots dx^3.$$

We compute

$$\frac{dV}{dt} = \theta V$$

so that

$$\frac{dV}{d\tau} = \theta V \frac{dt}{d\tau} = \theta V \frac{3}{\theta} = 3V.$$

Considering H as a function of τ , we get (27) using (17). \square

4. BIANCHI I

Bianchi I solutions of (5)-(7) are of the form $(\sigma_+, \sigma_-, 0, 0, 0)$ where (σ_+, σ_-) are independent of time and satisfy $\sigma_+^2 + \sigma_-^2 = 1$.

5. BIANCHI II

Proposition 1. *A Bianchi II solution of (5)-(7) with $N_1 > 0$ and $N_2 = N_3 = 0$ satisfies*

$$(28) \quad \lim_{\tau \rightarrow \infty} N_1 = 0$$

and

$$(29) \quad \lim_{\tau \rightarrow \infty} (\Sigma_+, \Sigma_-) = (\sigma_+, \sigma_-)$$

where (σ_+, σ_-) belongs to \mathcal{K}_1 .

Proof. Using the constraint (7) we deduce that

$$\Sigma'_+ = \frac{3}{2}N_1^2(2 - \Sigma_+).$$

We wish to apply the monotonicity principle. There are three variables. Let U be defined by $N_1 > 0$, M be defined by (7) and $F(\Sigma_+, \Sigma_-, N_1) = \Sigma_+$. Equation (28) follows, since if N_1 does not converge to zero, we can construct an ω -limit point in U due to the fact that the variables are contained in a compact set. Combining this with the constraint we deduce

$$\lim_{\tau \rightarrow \infty} q = 2.$$

Since Σ_+ is monotone and bounded, it converges. Since $q \rightarrow 2$ and the ω -limit set is connected, c.f. Lemma 1, (29) holds, but we do not yet know anything about the limit. Compute

$$\left(\frac{\Sigma_-}{2 - \Sigma_+} \right)' = 0.$$

We get

$$(30) \quad \frac{\Sigma_-}{2 - \Sigma_+} = \frac{\sigma_-}{2 - \sigma_+}$$

for arbitrary (Σ_+, Σ_-) belonging to the solution. Since $N'_1 = (q - 4\Sigma_+)N_1$ and $N_1 \rightarrow 0$ we have to have $\sigma_+ \geq \frac{1}{2}$. If $\sigma_+ = \frac{1}{2}$ then $\sigma_- = \pm \frac{\sqrt{3}}{2}$. The two corresponding lines in the $\Sigma_+\Sigma_-$ -plane obtained by substituting (σ_+, σ_-) into (30) do not intersect any points interior to the Kasner circle. Therefore $\sigma_+ = \frac{1}{2}$ is not an allowed limit point and the proposition follows. \square

6. BIANCHI VI₀

Proposition 2. *Consider a Bianchi VI₀ solution of (5)-(7) with $N_1 = 0$, $N_2 > 0$ and $N_3 < 0$. Then*

$$(31) \quad \lim_{\tau \rightarrow \infty} N_2 = 0, \quad \lim_{\tau \rightarrow \infty} N_3 = 0$$

and

$$(32) \quad \lim_{\tau \rightarrow \infty} \Sigma_- = 0, \quad \lim_{\tau \rightarrow \infty} \Sigma_+ = -1.$$

Proof. By the constraint (7) the variables belong to a compact set. Since

$$\Sigma'_+ = -(2 - q)(1 + \Sigma_+)$$

Σ_+ is strictly monotone as long as $q < 2$ which is true for the entire solution. The monotonicity principle yields

$$(33) \quad \lim_{\tau \rightarrow \infty} q = 2$$

for similar reasons as in the proof of Proposition 1. By the constraint we conclude that (31) holds. Since Σ_+ is monotone, (33) holds and the ω -limit set is connected, we have

$$\lim_{\tau \rightarrow \infty} (\Sigma_+, \Sigma_-) = (\sigma_+, \sigma_-)$$

with $\sigma_+^2 + \sigma_-^2 = 1$. Since N_2 and N_3 converge to zero and since they satisfy (5) we must have

$$\begin{aligned} 2(\sigma_+^2 + \sigma_-^2) + 2\sigma_+ + 2\sqrt{3}\sigma_- &\leq 0 \\ 2(\sigma_+^2 + \sigma_-^2) + 2\sigma_+ - 2\sqrt{3}\sigma_- &\leq 0. \end{aligned}$$

Adding these inequalities we obtain $4 + 4\sigma_+ \leq 0$. Equation (32) follows. \square

Let us now record the fact that Bianchi VIII is very different from Bianchi IX in the direction towards the singularity.

Proposition 3. *For every $\epsilon > 0$, there exists Bianchi VIII vacuum initial data y_ϵ and a real number $T_\epsilon > 0$, where y_ϵ is closer to the Kasner circle than ϵ , such that evolving y_ϵ a time T_ϵ to the past yields a Bianchi VIII point with the property that*

$$|N_1 N_2| + |N_2 N_3| + |N_3 N_1| \geq \frac{1}{4}.$$

Remark. By distance to the Kasner circle we here mean the Euclidean distance in \mathbb{R}^5 .

Proof. Let $\epsilon > 0$ and x be any Bianchi VI₀ initial data. By Proposition 2, there is a $T_\epsilon > 0$ such that applying the flow to x , $\Phi(T_\epsilon, x)$ is closer to the Kasner circle than $\epsilon/2$, where $\Phi(\tau, x)$ represents the solution to (5)-(7) with initial data x evaluated in τ . However, there are Bianchi VIII points as close to $\Phi(T_\epsilon, x)$ as we wish. Given an $\eta > 0$, we can thus, by the continuity of the flow, choose Bianchi VIII initial data y_ϵ with the property that y_ϵ is closer to the Kasner circle than ϵ and $\Phi(-T_\epsilon, y_\epsilon)$ is closer to x than η . Choosing x to have $\Sigma_+ = \Sigma_- = 0$ and $N_2 = -N_3 = \frac{1}{\sqrt{3}}$ and letting η be small enough, we get the conclusion of the proposition. \square

7. BIANCHI VII₀

The following proposition is a consequence of (5)-(7).

Proposition 4. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2 = N_3 > 0$, $N_1 = 0$ and $\Sigma_- = 0$. Then one of the following statements holds*

- $\Sigma_+ = 1$ for the entire solution and $N_2 = N_3 \rightarrow \infty$ as $\tau \rightarrow \infty$.
- $\Sigma_+ = -1$ and $N_2 = N_3$ are constant for the entire solution.

Given a solution to (5)-(7), if $N_2(\tau) = N_3(\tau)$ and $\Sigma_-(\tau) = 0$ for one τ then the two equalities hold for all τ . Thus the above proposition together with the one below exhaust the possibilities for Bianchi VII₀ solutions.

Proposition 5. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2, N_3 > 0$ and $N_1 = 0$ which never satisfies $N_2 = N_3$ and $\Sigma_- = 0$ simultaneously. Then*

$$\lim_{\tau \rightarrow \infty} \Sigma_+ = -1, \quad \lim_{\tau \rightarrow \infty} \Sigma_- = 0$$

and

$$\lim_{\tau \rightarrow \infty} (N_2 - N_3) = 0.$$

Remark. In Proposition 6, we also prove that N_2 and N_3 converge to finite positive values.

Proof. **Assume first that there is an ω -limit point.** Then the monotonicity principle suffices. Consider

$$(34) \quad \Sigma'_+ = -(2-q)(\Sigma_+ + 1).$$

We apply the monotonicity principle with $F(\Sigma_+, \Sigma_-, N_2, N_3) = \Sigma_+$, M defined by (7) and U defined by $\Sigma_-^2 + (N_2 - N_3)^2 > 0$ and $N_2 + N_3 > 0$. F is strictly monotone on a solution in $U \cap M$ since if $\Sigma'_+(\tau) = 0$ then $[N_2 - N_3](\tau) = 0$ so that $\Sigma_-(\tau) \neq 0$ whence $[N_2 - N_3]'(\tau) \neq 0$, c.f. (5)-(7). If $\tau_k \rightarrow \infty$ yields an ω -limit point, we must thus have $[\Sigma_-^2 + (N_2 - N_3)^2](\tau_k) \rightarrow 0$ or $(N_2 + N_3)(\tau_k) \rightarrow 0$ by the monotonicity principle. However

$$Z_{-1} = \frac{\frac{4}{3}\Sigma_-^2 + (N_2 - N_3)^2}{N_2 N_3} = \frac{\frac{4}{3}(1 - \Sigma_+^2)}{N_2 N_3}$$

has non-positive derivative, and consequently $[\Sigma_-^2 + (N_2 - N_3)^2](\tau_k) \rightarrow 0$ (if it does not converge to zero, $Z_{-1}(\tau_k) \rightarrow \infty$). We conclude that $\Sigma_+(\tau_k) \rightarrow -1$. By the monotonicity of Σ_+ the proposition follows.

Assume that there is no ω -limit point. If there is a sequence $\tau_k \rightarrow \infty$ such that $N_2(\tau_k)$ is bounded, then there is an ω -limit point by the constraint (7). Similarly if there is a sequence such that N_3 is bounded. We can thus assume that N_2 and N_3 converge to ∞ .

The variable Σ_+ is still decreasing and bounded. We can thus assume it to converge to σ_+ with $-1 \leq \sigma_+ < 1$. If $\sigma_+ = -1$ we are done so assume not. We prove that this assumption leads to the consequence that $(2-q) \notin L^1([0, \infty))$. Combining this with (34) we conclude that $\Sigma_+ \rightarrow -1$ contradicting our assumption.

The intuitive idea is as follows. $\Sigma_+ \rightarrow \sigma_+$ but Σ_- will oscillate between $\pm(1 - \sigma_+^2)^{1/2}$, roughly speaking. When Σ_- is small we get a contribution to the integral of $2 - q$. If Σ_- spends most of its time close to $\pm(1 - \sigma_+^2)^{1/2}$ we would have a problem, but that turns out not to be the case.

If N_2 and N_3 converge to infinity, we will see that the solution will become oscillatory. To be more precise, let x and y be defined by

$$(35) \quad x = \frac{\Sigma_-}{(1 - \Sigma_+^2)^{1/2}}$$

$$(36) \quad y = \frac{\sqrt{3}}{2} \frac{N_2 - N_3}{(1 - \Sigma_+^2)^{1/2}}.$$

Since $\Sigma_+^2(\tau) < 1$ for all τ , x and y are smooth. Let

$$(37) \quad g = 3(N_2 + N_3) + 2(1 + \Sigma_+)xy.$$

Then $x' = -gy$ and $y' = gx$. By the constraint $x^2 + y^2 = 1$ so that we can choose a ϕ_0 such that $(x(\tau_0), y(\tau_0)) = (\cos(\phi_0), \sin(\phi_0))$. Define

$$(38) \quad \xi(\tau) = \int_{\tau_0}^{\tau} g(s)ds + \phi_0.$$

Then $x(\tau) = \cos(\xi(\tau))$ and $y(\tau) = \sin(\xi(\tau))$.

As mentioned it suffices to prove that $2 - q \notin L^1([0, \infty))$ under the assumption that $-1 < \sigma_+ < 1$. Consider

$$2 - q = 2(1 - \Sigma_+^2 - (1 - \Sigma_+^2)\cos^2(\xi)) = 2(1 - \Sigma_+^2)\sin^2(\xi) \geq 2(1 - \sigma_+^2)\sin^2(\xi)$$

$$= c \sin^2(\xi),$$

where $c > 0$ by assumption. Consider a period. Assume T is great enough that $1 < 5(N_2 + N_3)/2 \leq g \leq 7(N_2 + N_3)/2$ for $\tau \geq T$ and that $\xi(\tau_2) - \xi(\tau_1) = 2\pi$ where $T \leq \tau_1 < \tau_2$. Let us estimate $\Delta\tau = \tau_2 - \tau_1$. By considering (38) we conclude that $\Delta\tau$ can be chosen arbitrarily small if T is great enough since $N_2, N_3 \rightarrow \infty$. Using (5) we derive the consequence that for every $\epsilon > 0$

$$1 - \epsilon \leq \frac{(N_2 + N_3)(\tau)}{(N_2 + N_3)(\tau_1)} \leq 1 + \epsilon$$

for every $\tau \in [\tau_1, \tau_2]$ if T is great enough. Thus

$$\Delta\tau = \int_{\tau_1}^{\tau_2} d\tau = \int_{\xi(\tau_1)}^{\xi(\tau_2)} \frac{1}{g} d\xi \leq \int_{\xi(\tau_1)}^{\xi(\tau_2)} \frac{2}{5(N_2 + N_3)} d\xi \leq \frac{8\pi}{5(N_2 + N_3)(\tau_1)}$$

if T is great enough. But then

$$\begin{aligned} \int_{\tau_1}^{\tau_2} (2 - q) d\tau &\geq \int_{\tau_1}^{\tau_2} c \sin^2(\xi) d\tau = \int_{\xi(\tau_1)}^{\xi(\tau_2)} c \sin^2(\xi) \frac{1}{g} d\xi \\ &\geq \int_{\xi(\tau_1)}^{\xi(\tau_2)} c \sin^2(\xi) \frac{2}{7(N_2 + N_3)} d\xi \geq \frac{c}{7(N_2 + N_3)(\tau_1)} \int_{\xi(\tau_1)}^{\xi(\tau_2)} \sin^2(\xi) d\xi \\ &= \frac{c\pi}{7(N_2 + N_3)(\tau_1)} \geq \frac{5c}{56} \Delta\tau = c' \Delta\tau \end{aligned}$$

if T is great enough. Here $c' > 0$. Let $M > 0$ be any number. Consider an interval $[\tau_a, \tau_b]$ where τ_a is great enough that the above conditions are met, $(\tau_b - \tau_a)/c' > M$ and $\xi(\tau_b) - \xi(\tau_a)$ is an integer multiple of 2π . Then

$$\int_{\tau_a}^{\tau_b} (2 - q) d\tau \geq M$$

and consequently $2 - q \notin L^1([0, \infty))$. The proposition follows. \square

It will be of interest to know that N_2 is bounded away from zero.

Lemma 6. *Consider a Bianchi VII₀ solution to (5)-(7). Then*

$$N_2 N_3 \geq c > 0$$

on $[0, \infty)$, where c is a positive constant.

Proof. Assume there is a time sequence $\tau_k \rightarrow \infty$ such that $(N_2 N_3)(\tau_k)$ converges to zero. Since $N_2 - N_3 \rightarrow 0$, we conclude that the solution evaluated at τ_k converges to the Kasner circle. Since the flow is continuous, and Bianchi VII₀ belongs to the boundary of Bianchi IX, we can construct a sequence of Bianchi IX initial data x_k converging to the Kasner circle, such that $\Phi(-\tau_k, x_k)$ converges to a Bianchi VII₀ point with $N_2 N_3 > 0$. This is impossible by Theorem 15.1 of [4]. \square

As was noted in the proof of Proposition 5, the objects x and y defined in (35) and (36) can be written as

$$x(\tau) = \cos \xi(\tau), \quad y(\tau) = \sin \xi(\tau)$$

where ξ is defined in (38). Since $N_2 + N_3$ is bounded away from zero to the future by Lemma 6 and $1 + \Sigma_+$ converges to zero, we can assume that

$$(39) \quad g(\tau) \geq 2(N_2 + N_3)(\tau) \geq c > 0 \quad \forall \tau \geq \tau_0$$

for some positive constant c , where g is defined by (37). In other words, the expressions x and y will carry out an infinite number of oscillations as time progresses. It is natural to average over a period of this oscillation in order to analyze the asymptotics, and we need to know how certain expressions vary over a period to be able to do this.

In the rest of this section we will implicitly assume that all Bianchi VII₀ solutions satisfy $\Sigma_-^2 + (N_2 - N_3)^2 > 0$.

Lemma 7. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2, N_3 > 0$, and let τ_0 be such that (39) is fulfilled. Let $\tau_0 \leq \tau_a < \tau_b$, and assume*

$$\xi(\tau_b) - \xi(\tau_a) = \pi.$$

Then there is a T such that

$$(40) \quad \left| 1 - \frac{(1 + \Sigma_+)(\tau_1)}{(1 + \Sigma_+)(\tau_2)} \right| \leq C \frac{(1 + \Sigma_+)(\tau_b)}{(N_2 + N_3)(\tau_{\max})}$$

and

$$(41) \quad \left| 1 - \frac{(N_2 + N_3)(\tau_1)}{(N_2 + N_3)(\tau_2)} \right| \leq C \frac{(1 + \Sigma_+)(\tau_b)}{(N_2 + N_3)(\tau_{\max})}$$

if $\tau_a \geq T$, where τ_{\max} yields the maximum value of $N_2 + N_3$ in $[\tau_a, \tau_b]$ and τ_1, τ_2 are arbitrary members of $[\tau_a, \tau_b]$. The constant C only depends on the constant c appearing in (39).

Proof. Observe that

$$\pi = \int_{\tau_a}^{\tau_b} g(s) ds \geq c(\tau_b - \tau_a),$$

so that

$$(42) \quad \tau_b - \tau_a \leq \frac{\pi}{c}.$$

Consider

$$(N_2 + N_3)' = (q + 2\Sigma_+)(N_2 + N_3) + 2\sqrt{3}\Sigma_-(N_2 - N_3).$$

Estimate

$$|q + 2\Sigma_+| \leq 2|\Sigma_+(1 + \Sigma_+)| + 2\Sigma_-^2 \leq 2(1 + \Sigma_+) + 2(1 - \Sigma_+^2) \leq 6(1 + \Sigma_+),$$

and

$$|2\sqrt{3}\Sigma_-(N_2 - N_3)| \leq 2[\Sigma_-^2 + \frac{3}{4}(N_2 - N_3)^2] \leq 2(1 - \Sigma_+^2) \leq 4(1 + \Sigma_+).$$

Since $N_2 + N_3$ is bounded below by a positive constant for $\tau \geq \tau_0$, we conclude that

$$|(N_2 + N_3)'| \leq C_1(1 + \Sigma_+)(N_2 + N_3)$$

for some positive constant C_1 . Letting $\tau_1, \tau_2 \in [\tau_a, \tau_b]$, we get

$$(43) \quad \ln \alpha = \ln \left(\frac{(N_2 + N_3)(\tau_1)}{(N_2 + N_3)(\tau_2)} \right) \leq C_1 \int_{\tau_a}^{\tau_b} (1 + \Sigma_+) ds = \beta,$$

where we have introduced the quantities α and β in order to make the following arguments easier to follow. If $\alpha \geq 1$, then

$$(44) \quad 0 \leq \alpha - 1 \leq e^\beta - 1 = \sum_{k=1}^{\infty} \frac{\beta^k}{k!} \leq \beta e^\beta.$$

If $\alpha \leq 1$, then (44) also holds if we replace α with $1/\alpha$, since τ_1 and τ_2 in (43) are arbitrary. Multiplying this inequality with α then yields the conclusion

$$|1 - \alpha| \leq \beta e^\beta$$

without any conditions on α . Since we have the bound (42) and $0 \leq 1 + \Sigma_+ \leq 2$, we conclude that e^β is bounded by a constant, so that

$$(45) \quad \left| 1 - \frac{(N_2 + N_3)(\tau_1)}{(N_2 + N_3)(\tau_2)} \right| \leq C_2 \int_{\tau_a}^{\tau_b} (1 + \Sigma_+) ds.$$

Observe that this inequality proves that the left hand side can be chosen to be arbitrarily small by demanding τ_a to be big enough. Let us estimate the integral on the right hand side. Assuming τ_a to be great enough that the right hand side of (45) is less than $\frac{1}{2}$, and using (39) and the fact that Σ_+ is monotone, we conclude that

$$\begin{aligned} \int_{\tau_a}^{\tau_b} (1 + \Sigma_+) ds &\leq (1 + \Sigma_+)(\tau_a) \int_{\xi(\tau_a)}^{\xi(\tau_b)} \frac{1}{g} d\xi \leq (1 + \Sigma_+)(\tau_a) \int_{\xi(\tau_a)}^{\xi(\tau_b)} \frac{1}{2(N_2 + N_3)} d\xi \\ &\leq \frac{3\pi}{4} \frac{(1 + \Sigma_+)(\tau_a)}{(N_2 + N_3)(\tau_{\max})} \end{aligned}$$

where τ_{\max} corresponds to the maximum value of $N_2 + N_3$ in $[\tau_1, \tau_2]$. Combining this with (45) we get

$$(46) \quad \left| 1 - \frac{(N_2 + N_3)(\tau_1)}{(N_2 + N_3)(\tau_2)} \right| \leq C_3 \frac{(1 + \Sigma_+)(\tau_a)}{(N_2 + N_3)(\tau_{\max})}.$$

Consider now

$$(47) \quad (1 + \Sigma_+)' = -(2 - q)(1 + \Sigma_+)$$

Since

$$0 \leq 2 - q \leq 4(1 + \Sigma_+),$$

we conclude that

$$(48) \quad \left| 1 - \frac{(1 + \Sigma_+)(\tau_1)}{(1 + \Sigma_+)(\tau_2)} \right| \leq C_3 \frac{(1 + \Sigma_+)(\tau_a)}{(N_2 + N_3)(\tau_{\max})}$$

by an argument similar to that proving (46). By assuming τ_a to be big enough (48) leads to the conclusion that we, at the cost of increasing the constants, can replace $(1 + \Sigma_+)(\tau_a)$ in the right hand side of (46) and (48) with the same expression evaluated in τ_b , yielding the statement of the lemma. \square

The following lemma constitutes the main observation.

Lemma 8. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2, N_3 > 0$, and let τ_0 be such that (39) is fulfilled. Let $\tau_0 \leq \tau_a < \tau_b$, and assume*

$$\xi(\tau_b) - \xi(\tau_a) = \pi.$$

Then there is a T such that

$$(49) \quad \left| \int_{\tau_a}^{\tau_b} [\Sigma_-^2 - (1 + \Sigma_+)] ds \right| \leq C \int_{\tau_a}^{\tau_b} (1 + \Sigma_+)^2 ds$$

if $\tau_a \geq T$, for some constant C depending only on the constant c appearing in (39).

Proof. Observe first that

$$\Sigma_-^2 = (1 - \Sigma_+^2)x^2 = (1 - \Sigma_+)(1 + \Sigma_+)x^2 = 2(1 + \Sigma_+)x^2 - (1 + \Sigma_+)^2x^2.$$

In consequence, the integrand of interest is

$$2(1 + \Sigma_+)x^2 - (1 + \Sigma_+) = (1 + \Sigma_+) \cos \eta$$

where $\eta(\tau) = 2\xi(\tau)$. Let $\eta_a = \eta(\tau_a)$ and $\eta_b = \eta(\tau_b) = \eta_a + 2\pi$ and observe that

$$\begin{aligned} \int_{\tau_a}^{\tau_b} (1 + \Sigma_+) \cos \eta d\tau &= \int_{\eta_a}^{\eta_b} \frac{1 + \Sigma_+}{2g} \cos \eta d\eta \\ &= \int_{\eta_a}^{\eta_a + \pi} \left(\frac{1 + \Sigma_+(\eta)}{2g(\eta)} - \frac{1 + \Sigma_+(\eta + \pi)}{2g(\eta + \pi)} \right) \cos \eta d\eta \end{aligned}$$

where we in order to obtain the last equality divided the integral into one part from η_a to $\eta_a + \pi$ and one from $\eta_a + \pi$ to $\eta_a + 2\pi$, and then made the substitution $\eta \rightarrow \eta - \pi$ in the second integral. We now wish to prove that the absolute value of the integrand in the integral from η_a to $\eta_a + \pi$ can be bounded by

$$C_1 \frac{[1 + \Sigma_+(\eta)]^2}{2g(\eta)} + C_2 \frac{[1 + \Sigma_+(\eta + \pi)]^2}{2g(\eta + \pi)},$$

since this would prove the assertion. Let us first rewrite

$$\frac{1 + \Sigma_+(\eta)}{2g(\eta)} - \frac{1 + \Sigma_+(\eta + \pi)}{2g(\eta + \pi)} = \frac{1 + \Sigma_+(\eta)}{2g(\eta)} \left(1 - \frac{g(\eta)}{g(\eta + \pi)} \right) - \frac{\Sigma_+(\eta + \pi) - \Sigma_+(\eta)}{2g(\eta + \pi)}.$$

Using (40) and the fact that $N_2 + N_3$ is bounded from below by a positive constant, we note that the second term can be estimated in the desired way; multiply (40) by $1 + \Sigma_+(\tau_2)$ in order to get an estimate of $\Sigma_+(\tau_1) - \Sigma_+(\tau_2)$. If we had

$$1 - \frac{(N_2 + N_3)(\eta)}{(N_2 + N_3)(\eta + \pi)}$$

instead of

$$1 - \frac{g(\eta)}{g(\eta + \pi)}$$

in the first term, then we could estimate the first term in the desired way as well. However, using (40), (41) and the fact that g and $N_2 + N_3$ can be bounded from below by a positive constant, one can show that

$$\frac{(N_2 + N_3)(\eta)}{(N_2 + N_3)(\eta + \pi)} - \frac{g(\eta)}{g(\eta + \pi)}$$

is small enough to yield the desired bound. \square

Lemma 9. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2, N_3 > 0$, and let τ_0 be such that (39) is fulfilled. Then*

$$(50) \quad \lim_{\tau \rightarrow \infty} \frac{\int_{\tau_0}^{\tau} \Sigma_-^2 ds}{\int_{\tau_0}^{\tau} (1 + \Sigma_+) ds} = 1$$

and

$$(51) \quad \lim_{\tau \rightarrow \infty} \frac{\int_{\tau_0}^{\tau} (2 - q) ds}{\int_{\tau_0}^{\tau} 2(1 + \Sigma_+) ds} = 1.$$

Proof. Observe first that since (47) is fulfilled and $1 + \Sigma_+ \rightarrow 0$ as $\tau \rightarrow \infty$, $2 - q \notin L^1([\tau_0, \infty))$. Moreover,

$$2 - q \leq 2(1 - \Sigma_+^2) \leq 4(1 + \Sigma_+),$$

so that $1 + \Sigma_+ \notin L^1([\tau_0, \infty))$. In order to prove (50), let us prove that

$$\lim_{\tau \rightarrow \infty} \frac{\int_{\tau_0}^{\tau} [\Sigma_-^2 - (1 + \Sigma_+)] ds}{\int_{\tau_0}^{\tau} (1 + \Sigma_+) ds} = 0.$$

Let $\epsilon > 0$ and let T be such that Lemma 8 can be applied and such that $1 + \Sigma_+(\tau) \leq \epsilon/(2C)$ for $\tau \geq T$, where C is the constant appearing in (49). Given $\tau > T$, let τ_1 be the greatest number smaller than τ such that $\xi(\tau_1) - \xi(T)$ is an integer multiple of π . We have

$$\begin{aligned} \left| \int_{\tau_0}^{\tau} [\Sigma_-^2 - (1 + \Sigma_+)] ds \right| &\leq \int_{\tau_0}^T |\Sigma_-^2 - (1 + \Sigma_+)| ds + \left| \int_T^{\tau_1} [\Sigma_-^2 - (1 + \Sigma_+)] ds \right| \\ &\quad + \int_{\tau_1}^{\tau} |\Sigma_-^2 - (1 + \Sigma_+)| ds. \end{aligned}$$

We should divide this expression with $\int_{\tau_0}^{\tau} (1 + \Sigma_+) ds$, and then evaluate the limit as $\tau \rightarrow \infty$. After having carried out this division, the first and the third terms will yield something that can be assumed to be arbitrarily small by assuming τ to be big enough. Using (49), the middle term can be bounded by

$$\begin{aligned} \left| \int_T^{\tau_1} [\Sigma_-^2 - (1 + \Sigma_+)] ds \right| &\leq C \int_T^{\tau_1} (1 + \Sigma_+)^2 ds \leq C[1 + \Sigma_+(T)] \int_T^{\tau_1} (1 + \Sigma_+) ds \\ &\leq \frac{\epsilon}{2} \int_T^{\tau_1} (1 + \Sigma_+) ds. \end{aligned}$$

After having carried out the division, the corresponding contribution can consequently be assumed to be less than or equal to $\epsilon/2$. Equation (50) follows. In order to prove (51), observe that

$$(2 - q) - 2(1 + \Sigma_+) = 2(1 + \Sigma_+) - 2\Sigma_-^2 - 2(1 + \Sigma_+)^2.$$

In consequence the proof of (51) is similar to the proof of (50). \square

Lemma 10. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2, N_3 > 0$, and let τ_0 be such that (39) is fulfilled. Then, if $p > 1$,*

$$1 + \Sigma_+ \in L^p([\tau_0, \infty)).$$

Proof. By (51) we conclude that for each $\alpha_1 < 2 < \alpha_2$, there is a T such that $\tau \geq T$ implies

$$\alpha_1 < \frac{\int_{\tau_0}^{\tau} (2 - q) ds}{\int_{\tau_0}^{\tau} (1 + \Sigma_+) ds} < \alpha_2.$$

Introduce the notation

$$\alpha = 1 + \Sigma_+(\tau_0)$$

and

$$h(\tau) = \int_{\tau_0}^{\tau} [1 + \Sigma_+(s)] ds.$$

By (47) we conclude that

$$(52) \quad \alpha \exp[-\alpha_2 h(\tau)] \leq h'(\tau) \leq \alpha \exp[-\alpha_1 h(\tau)]$$

for all $\tau \geq T$. Integrating the leftmost inequality, we get

$$\frac{1}{\alpha_2} \ln[\exp[\alpha_2 h(T)] + \alpha \alpha_2 (\tau - T)] \leq h(\tau)$$

for all $\tau \geq T$. Inserting this in the rightmost inequality of (52), we conclude that

$$h'(\tau) \leq \alpha [\exp[\alpha_2 h(T)] + \alpha \alpha_2 (\tau - T)]^{-\frac{\alpha_1}{\alpha_2}}.$$

Since $h' = 1 + \Sigma_+$, and since the quotient α_1/α_2 can be chosen to be arbitrarily close to one, the lemma follows. \square

Proposition 6. *Consider a Bianchi VII₀ solution to (5)-(7) with $N_2, N_3 > 0$. Then*

$$\lim_{\tau \rightarrow \infty} N_2(\tau) = \lim_{\tau \rightarrow \infty} N_3(\tau) = n_0$$

where $0 < n_0 < \infty$.

Proof. It is enough to prove that $N_2 N_3$ converges to a positive number. The reason is that this would prove that all the variables are contained in a compact set to the future, so that

$$N_2 = (N_2 N_3)^{1/2} + N_2^{1/2} \frac{N_2 - N_3}{N_2^{1/2} + N_3^{1/2}}$$

converges, since $N_2 N_3$ converges, $N_2^{1/2} + N_3^{1/2}$ is bounded from below by a positive constant, $N_2^{1/2}$ is bounded from above, and $N_2 - N_3 \rightarrow 0$ by Proposition 5. Since $N_2 - N_3 \rightarrow 0$, the limit for N_3 follows. Since

$$(N_2 N_3)' = (2q + 4\Sigma_+) N_2 N_3,$$

the essential point is to prove that

$$\int_{\tau_0}^{\tau} (q/2 + \Sigma_+) ds$$

converges as $\tau \rightarrow \infty$. Observe now that

$$q/2 + \Sigma_+ = \Sigma_+(1 + \Sigma_+) + \Sigma_-^2 = (1 + \Sigma_+)^2 - (1 + \Sigma_+) + \Sigma_-^2.$$

Since $1 + \Sigma_+ \in L^2([\tau_0, \infty))$ by Lemma 10, we need only consider

$$\psi(\tau) = \int_{\tau_0}^{\tau} [\Sigma_-^2 - (1 + \Sigma_+)] ds.$$

Let T be big enough that (49) is applicable. If $\tau_2 \geq \tau_1 \geq T$ and τ_3 is the largest number smaller than τ_2 such that

$$\xi(\tau_3) - \xi(\tau_1)$$

is an integer multiple of π , then

$$\psi(\tau_2) - \psi(\tau_1) = \int_{\tau_1}^{\tau_3} [\Sigma_-^2 - (1 + \Sigma_+)] ds + \int_{\tau_3}^{\tau_2} [\Sigma_-^2 - (1 + \Sigma_+)] ds.$$

The second integral converges to zero as τ_1 and τ_2 go to infinity, since the integrand goes to zero and the length of the interval is bounded (42). By (49) we conclude that

$$\left| \int_{\tau_1}^{\tau_3} [\Sigma_-^2 - (1 + \Sigma_+)] ds \right| \leq C \int_{\tau_1}^{\tau_3} (1 + \Sigma_+)^2 ds.$$

Since $1 + \Sigma_+ \in L^2([\tau_0, \infty))$, this expression converges to zero as τ_1 goes to infinity. The lemma follows. \square

8. BIANCHI VIII

We assume that $N_2, N_3 > 0$ and $N_1 < 0$. Let us begin by giving the asymptotic behaviour of the N_i .

Lemma 11. *Consider a Bianchi VIII solution to (5)-(7) with $N_2, N_3 > 0$ and $N_1 < 0$. Then,*

$$\lim_{\tau \rightarrow \infty} N_1(\tau) = 0, \quad \lim_{\tau \rightarrow \infty} N_2(\tau) = \lim_{\tau \rightarrow \infty} N_3(\tau) = \infty.$$

Proof. Reformulate the constraint (7) as

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + (N_2 - N_3)^2 - 2N_1(N_2 + N_3)] = 1.$$

Observe that all terms appearing on the left hand side are non-negative so that the absolute values of, for instance, N_1 and $N_2 - N_3$ are bounded by numerical constants. Assume there is a sequence $\tau_k \rightarrow \infty$ such that $N_2(\tau_k) \leq C < \infty$. Then the $N_i(\tau_k)$ are uniformly bounded. By the constraint Σ_+ and Σ_- are contained in a compact set. Thus we can choose a convergent subsequence so that we have an ω -limit point $(\sigma_+, \sigma_-, n_1, n_2, n_3)$. We apply the monotonicity principle with U defined by $N_i \neq 0$, $i = 1, 2, 3$, M defined by (7) and $F(\Sigma_+, \Sigma_-, N_1, N_2, N_3) = N_1 N_2 N_3$. F , evaluated on a solution contained in $U \cap M$, is strictly monotone since $(N_1 N_2 N_3)' = 3q N_1 N_2 N_3$. If $q(\tau) = 0$ then either $\Sigma'_-(\tau) \neq 0$ or $\Sigma'_+(\tau) \neq 0$ by (5)-(7). By the monotonicity principle one of the n_i must be zero contradicting the growth of $|N_1 N_2 N_3|$. We conclude that $N_2 \rightarrow \infty$, but then $N_3 \rightarrow \infty$ and $N_1 \rightarrow 0$ since $N_1(N_2 + N_3)$ and $N_2 - N_3$ are bounded by the constraint (7). \square

8.1. The NUT case. Consider the special case $N_2 = N_3$ and $\Sigma_- = 0$.

Proposition 7. *A Bianchi VIII solution with $N_1 < 0$, satisfying $N_2 = N_3 > 0$ and $\Sigma_- = 0$ has the following asymptotic behaviour*

$$\lim_{\tau \rightarrow \infty} \Sigma_+(\tau) = \frac{1}{2}, \quad \lim_{\tau \rightarrow \infty} N_1(\tau) = 0, \quad \lim_{\tau \rightarrow \infty} N_2(\tau) = \infty \quad \text{and} \quad \lim_{\tau \rightarrow \infty} (N_1 N_2)(\tau) = -\frac{1}{4}.$$

Proof. The constraint implies

$$\Sigma'_+ = (1 - \Sigma_+^2)(1 - 2\Sigma_+) + \frac{9}{4}N_1^2.$$

Observe that we cannot have $\Sigma_+ \rightarrow 1$ since $N_1 N_2$ diverges to infinity there, c.f. (5). Also, $\Sigma_+ \rightarrow -1$ is an impossibility since Σ'_+ is always positive when Σ_+ is close to -1 . Since $N_1 \rightarrow 0$ by Lemma 11 and $-1 < \Sigma_+(\tau) < 1 \forall \tau$ by the constraint (7) $\Sigma_+ \rightarrow \frac{1}{2}$ follows. The last limit follows from the first and the constraint. \square

8.2. Oscillatory behaviour. The behaviour of a Bianchi VIII solution as $\tau \rightarrow \infty$ is in some sense oscillatory. The quantities that oscillate are Σ_- and $N_2 - N_3$. In order to analyze the asymptotics, we quantify this behaviour. Let

$$(53) \quad \tilde{x} = \Sigma_-, \quad \tilde{y} = \frac{\sqrt{3}}{2}(N_2 - N_3).$$

We have

$$\tilde{x}' = -3(N_2 + N_3)\tilde{y} + \epsilon_x, \quad \epsilon_x = -(2 - q)\Sigma_- + 3N_1\tilde{y}.$$

Furthermore,

$$\tilde{y}' = 3(N_2 + N_3)\tilde{x} + \epsilon_y, \quad \epsilon_y = (q + 2\Sigma_+)\tilde{y}.$$

Observe that by the constraint (7) $|\tilde{x}|, |\tilde{y}|, |\epsilon_x|$ and $|\epsilon_y|$ are bounded by numerical constants whereas $N_2 + N_3 \rightarrow \infty$ by Lemma 11. Let $g = 3(N_2 + N_3)$,

$$A = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}$$

$\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})^t$ and $\epsilon = (\epsilon_x, \epsilon_y)^t$ so that $\tilde{\mathbf{x}}' = A\tilde{\mathbf{x}} + \epsilon$. Let

$$(54) \quad \xi(\tau) = \int_{\tau_0}^{\tau} g(s)ds + \phi_0$$

for some ϕ_0 and $x_1(\tau) = \cos(\xi(\tau))$, $y_1(\tau) = \sin(\xi(\tau))$. Then, if $\mathbf{x}_1 = (x_1, y_1)^t$, $\mathbf{x}'_1 = A\mathbf{x}_1$. Define

$$\Phi = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}.$$

Then $\Phi' = -A\Phi$ and $[A, \Phi] = 0$. Let

$$(55) \quad r(\tau) = (\tilde{x}^2(\tau) + \tilde{y}^2(\tau))^{1/2}$$

and

$$(56) \quad \mathbf{x}(\tau) = (x(\tau), y(\tau))^t = (r(\tau_0) \cos[\xi(\tau)], r(\tau_0) \sin[\xi(\tau)])^t$$

where ϕ_0 has been chosen so that $\mathbf{x}(\tau_0) = \tilde{\mathbf{x}}(\tau_0)$. Since $[\Phi(\mathbf{x} - \tilde{\mathbf{x}})]' = -\Phi\epsilon$ and $\Phi(\tau) \in SO(2) \forall \tau$ we have

$$\|\tilde{\mathbf{x}}(\tau) - \mathbf{x}(\tau)\| \leq \left| \int_{\tau_0}^{\tau} \|\epsilon(s)\| ds \right|.$$

By the constraint (7), we have $\|\epsilon\| \leq 9$, and thus

$$(57) \quad \|\tilde{\mathbf{x}}(\tau) - \mathbf{x}(\tau)\| \leq 9|\tau - \tau_0|.$$

The next lemma collects some technical estimates. We present them here in order not to interrupt the flow of later proofs.

Lemma 12. *Consider a Bianchi VIII solution of (5)-(7) and let C be a constant. Then there is a T depending on C and the initial values such that if $[\tau_1, \tau_2]$ is a time interval with $\tau_1 \geq T$ and*

$$|\tau_2 - \tau_1| \leq \frac{C}{(N_2 + N_3)(\tau_3)}$$

for some $\tau_3 \in [\tau_1, \tau_2]$, then

$$(58) \quad |\Sigma_+(t_1) - \Sigma_+(t_2)| \leq \frac{C_1}{(N_2 + N_3)(t_3)}$$

$$(59) \quad |[N_1(N_2 + N_3)](t_1) - [N_1(N_2 + N_3)](t_2)| \leq \frac{C_2}{(N_2 + N_3)(t_3)}$$

$$(60) \quad \left| 1 - \frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)} \right| \leq \frac{C_3}{(N_2 + N_3)(t_3)}$$

for arbitrary $t_1, t_2, t_3 \in [\tau_1, \tau_2]$ where C_1 , C_2 and C_3 are constants only depending on C .

Proof. By Lemma 11 we can assume T to be great enough that $\Delta\tau = \tau_2 - \tau_1 \leq 1$. The inequality

$$\left| \frac{(N_2 + N_3)'}{N_2 + N_3} \right| \leq 8$$

follows from (5)-(7). Thus

$$\left| \frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)} \right| \leq e^{8(\tau_2 - \tau_1)}$$

so that if $(N_2 + N_3)(t_1) \geq (N_2 + N_3)(t_2)$,

$$(61) \quad \left| 1 - \frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)} \right| \leq |e^{8\Delta\tau} - 1| \leq 8e^8\Delta\tau$$

since $\Delta\tau \leq 1$. Multiplying this inequality by $(N_2 + N_3)(t_2)/(N_2 + N_3)(t_1) \leq 1$ we conclude that the assumption $(N_2 + N_3)(t_1) \geq (N_2 + N_3)(t_2)$ is not essential. There is thus a constant $c < \infty$ such that

$$\frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)} \leq c$$

for any $t_1, t_2 \in [\tau_1, \tau_2]$. Consequently

$$(62) \quad \Delta\tau \leq \frac{C}{(N_2 + N_3)(\tau_3)} \leq \frac{cC}{(N_2 + N_3)(t_3)}.$$

By (5)-(7) $|\Sigma'_+|$ and $|[N_1(N_2 + N_3)]'|$ are bounded by constants independent of Bianchi VIII solution. Equations (58) and (59) follow from (62). Equation (60) follows from (61) and (62). \square

Let us give the intuitive idea behind the next lemma. How the variables vary during a period is not so interesting, what is interesting is to compute the change over a period. In that way the continuous time evolution is replaced with a discrete evolution. To make the time step independent of approximation we wish to see what happens from $\tilde{x} = 0$ to the next time $\tilde{x} = 0$, but then we need to know that we have such zeros of \tilde{x} . In order to be able to prove this we assume that r (55) is not too small. The assumption is not too restrictive; our ultimate goal is to prove that $r \rightarrow 0$ and that a general Bianchi VIII solution in some sense converges to a NUT solution. If r is big our iteration is well defined and we will use it to prove that r decreases and if r is already small we do not need it.

Lemma 13. *Consider a Bianchi VIII solution of (5)-(7). There is a T depending on the initial values such that if $\tau_0 \geq T$ and $r(\tau_0) \geq [(N_2 + N_3)(\tau_0)]^{-1/2}$, then \tilde{x} has at least four zeros $\tau_a < \tau_b \leq \tau_0 \leq \tau_c < \tau_d$ in*

$$[\tau_1, \tau_2] = [\tau_0 - \frac{\pi}{(N_2 + N_3)(\tau_0)}, \tau_0 + \frac{\pi}{(N_2 + N_3)(\tau_0)}]$$

such that \tilde{x} is non zero in (τ_a, τ_b) , (τ_b, τ_c) and (τ_c, τ_d) . Furthermore, for any two consecutive zeros, for example τ_a and τ_b , we have

$$(63) \quad |\xi(\tau_b) - \xi(\tau_a) - \pi| \leq \frac{C}{[(N_2 + N_3)(\tau_a)]^{1/2}}$$

where C is a numerical constant.

Proof. Let T be great enough that if $\tau \in [\tau_1, \tau_2]$ then

$$(64) \quad 3(N_2 + N_3)(\tau) \geq 5(N_2 + N_3)(\tau_0)/2 \geq 5(N_2 + N_3)(\tau)/4.$$

This is possible by Lemma 12 and Lemma 11. Let \mathbf{x} be as in (56) where $\phi_0 \in [0, 2\pi]$ in (54) has been chosen so that $\mathbf{x}(\tau_0) = \tilde{\mathbf{x}}(\tau_0)$. If $\tau \in [\tau_1, \tau_2]$ then

$$|\xi(\tau) - \xi(\tau_0)| = \left| \int_{\tau_0}^{\tau} g(s) ds \right| \geq \frac{5}{2}(N_2 + N_3)(\tau_0)|\tau - \tau_0|.$$

Since ξ is a monotonically growing function we conclude

$$[0, 2\pi] \subseteq [\phi_0 - \frac{5}{2}\pi, \phi_0 + \frac{5}{2}\pi] \subseteq [\xi(\tau_1), \xi(\tau_2)].$$

We pick out the zeros in the interval $[0, 2\pi]$. Let $\xi(\tau_\alpha) = 0$, $\xi(\tau_\beta) = \pi/4$, $\xi(\tau_\gamma) = 3\pi/4$, $\xi(\tau_\delta) = 5\pi/4$, $\xi(\tau_\epsilon) = 7\pi/4$ and $\xi(\tau_\zeta) = 2\pi$, where τ_α and so on belong to $[\tau_1, \tau_2]$. In the intervals $[\tau_\alpha, \tau_\beta]$, $[\tau_\gamma, \tau_\delta]$ and $[\tau_\epsilon, \tau_\zeta]$ \tilde{x} cannot be zero if T is great enough because of (57) and the fact that $r(\tau_0) \geq [(N_2 + N_3)(\tau_0)]^{-1/2}$. For the same reason \tilde{x}' cannot be zero in $[\tau_\beta, \tau_\gamma]$ and $[\tau_\delta, \tau_\epsilon]$ since $\tilde{x}' = -g\tilde{y} + \epsilon_x$. Thus there is exactly one zero in $[\tau_\beta, \tau_\gamma]$ and one in $[\tau_\delta, \tau_\epsilon]$ and no zeros in between. The remaining zeros are picked out in the same way. By the above it follows that for two consecutive zeros, for example τ_a and τ_b , $\pi/2 \leq \xi(\tau_b) - \xi(\tau_a) \leq 3\pi/2$. Thus

$$|\xi(\tau_b) - \xi(\tau_a) - \pi| \leq \frac{\pi}{2} |\sin(\xi(\tau_b) - \xi(\tau_a) - \pi)| \leq \frac{\pi}{2} (|\cos(\xi(\tau_a))| + |\cos(\xi(\tau_b))|)$$

$$\leq \frac{\pi}{2r(\tau_0)} (|\tilde{x}(\tau_a) - x(\tau_a)| + |\tilde{x}(\tau_b) - x(\tau_b)|) \leq \frac{9\pi^2}{[(N_2 + N_3)(\tau_0)]^{1/2}}$$

$$\leq \frac{18\pi^2}{[(N_2 + N_3)(\tau_a)]^{1/2}}$$

using (57) and (64). \square

Lemma 14. *Consider a Bianchi VIII solution of (5)-(7). There is a T such that if τ_a and τ_b are two consecutive zeros of \tilde{x} and $r(\tau_a) \geq [(N_2 + N_3)(\tau_a)]^{-1/2}$, $\tau_a \geq T$, then if z_1 and z_2 are Σ_+ evaluated in τ_a and τ_b respectively and w_1 and w_2 are $-N_1(N_2 + N_3)$ evaluated in τ_a and τ_b respectively, we have*

$$(65) \quad z_2 - z_1 = [-(1 - z_1)(1 + z_1)^2 + \frac{3}{2}w_1(2 - z_1)](\tau_b - \tau_a) + \epsilon_1$$

and

$$(66) \quad w_2 - w_1 = (2z_1^2 - 2z_1 + 2 - 3w_1)w_1(\tau_b - \tau_a) + \epsilon_2$$

where

$$(67) \quad |\epsilon_i| \leq \frac{C}{(N_2 + N_3)^{3/2}(\tau_a)}$$

for $i = 1, 2$ and some numerical constant C .

Remark. We will prove that $\tau_b - \tau_a$ is of the order of magnitude $1/(N_2 + N_3)$. Consequently we can ignore the error terms in (65) and (66) as long as the polynomial expressions appearing in front of $\tau_b - \tau_a$ are of the order of magnitude 1.

Proof. The idea is to consider the derivatives of Σ_+ and $-N_1(N_2 + N_3)$ and to integrate between τ_a and τ_b . We will see that Σ_+ and $-N_1(N_2 + N_3)$ vary slowly so that we can replace them with constants, up to an error we estimate. However, Σ_- and $N_2 - N_3$ do not vary slowly so that we will have to use (57) and to estimate $\int_{\tau_a}^{\tau_b} \sin^2(\xi(\tau)) d\tau$.

All numerical constants below will be denoted C . Observe that

$$(68) \quad |\tau_b - \tau_a| \leq \frac{C}{(N_2 + N_3)(\tau_a)}$$

by the arguments presented in the proof of Lemma 13, assuming T to be great enough. We will assume T is great enough that $3(N_2 + N_3)(\tau) \geq 5(N_2 + N_3)(\tau_0)/2 \geq (N_2 + N_3)(\tau)$ for all $\tau \in [\tau_a, \tau_b]$, as is possible by Lemma 12. We begin by analyzing the variation of some relevant expressions in $[\tau_a, \tau_b]$. By Lemma 12 we conclude

$$(69) \quad |\Sigma_+(\tau) - \Sigma_+(\tau_a)| \leq \frac{C}{(N_2 + N_3)(\tau_a)}$$

for all $\tau \in [\tau_a, \tau_b]$ and similarly for $N_1(N_2 + N_3)$. Assume $N_2 + N_3$ has a max in τ_{\max} in $[\tau_a, \tau_b]$ and a min in τ_{\min} in the same interval. Then

$$(70) \quad \left| \frac{1}{(N_2 + N_3)(\tau_{\max})} - \frac{1}{(N_2 + N_3)(\tau_{\min})} \right| \leq \frac{1}{(N_2 + N_3)(\tau_{\max})} \left| 1 - \frac{(N_2 + N_3)(\tau_{\max})}{(N_2 + N_3)(\tau_{\min})} \right| \leq \frac{C}{(N_2 + N_3)^2(\tau_a)}$$

by (60). Since ξ as defined in (54) is a diffeomorphism we can consider functions of τ as functions of ξ . We will be interested in the following integral

$$\begin{aligned} \int_{\tau_a}^{\tau_b} \sin^2 \xi(\tau) d\tau &= \int_{\xi(\tau_a)}^{\xi(\tau_b)} \sin^2(\xi) \frac{1}{g} d\xi = \frac{1}{g(\tau_a)} \int_{\xi(\tau_a)}^{\xi(\tau_a) + \pi} \sin^2(\xi) d\xi \\ &+ \frac{1}{g(\tau_a)} \int_{\xi(\tau_a) + \pi}^{\xi(\tau_b)} \sin^2(\xi) d\xi + \int_{\xi(\tau_a)}^{\xi(\tau_b)} \sin^2(\xi) \left(\frac{1}{g} - \frac{1}{g(\tau_a)} \right) d\xi \\ &= \frac{\pi}{2g(\tau_a)} + \delta_1 \end{aligned}$$

where

$$|\delta_1| \leq \frac{C}{[(N_2 + N_3)(\tau_a)]^{3/2}}$$

due to (63) and (70). Furthermore

$$(71) \quad \tau_a - \tau_b = \int_{\tau_a}^{\tau_b} d\tau = \int_{\xi(\tau_a)}^{\xi(\tau_b)} \frac{1}{g} d\xi = \frac{\pi}{g(\tau_a)} + \delta_2,$$

by similar arguments, where δ_2 is of the same order of magnitude as δ_1 . Consequently

$$(72) \quad \int_{\tau_a}^{\tau_b} \sin^2 \xi(\tau) d\tau = \frac{1}{2}(\tau_b - \tau_a) + \delta_3$$

where δ_3 is of the same order of magnitude as δ_1 . We now have all the necessary estimates at our disposal. In $[\tau_a, \tau_b]$ we have

$$\begin{aligned} \Sigma'_+ &= -(2-q)\Sigma_+ - 3S_+ = -(2-2\Sigma_+^2)\Sigma_+ + 2\Sigma_-^2\Sigma_+ - \frac{3}{2}(N_2 - N_3)^2 + 3N_1^2 \\ &\quad - \frac{3}{2}N_1(N_2 + N_3) = -(2-2z_1^2)z_1 + 2z_1r^2(\tau_a)\cos^2(\xi) - 2r^2(\tau_a)\sin^2(\xi) + \frac{3}{2}w_1 + \epsilon_3 \end{aligned}$$

where

$$|\epsilon_3| \leq \frac{C}{(N_2 + N_3)(\tau_a)}$$

due to (69), the analogous estimate for $N_1(N_2 + N_3)$, (57) and the fact that N_1 is of the order of magnitude $(N_2 + N_3)^{-1}$ (7). Integrating, using (72), we have

$$z_2 - z_1 = [-(2-2z_1^2)z_1 + z_1r^2(\tau_a) - r^2(\tau_a) + \frac{3}{2}w_1](\tau_b - \tau_a) + \epsilon_4$$

where

$$|\epsilon_4| \leq \frac{C}{(N_2 + N_3)^{3/2}(\tau_a)}.$$

Using the constraint (7) to express $r^2(\tau_a)$ in z_1 and w_1 (and N_1), we deduce (65) and (67), $i = 1$. The argument to obtain (66) is similar. \square

Lemma 15. *Consider a Bianchi VIII solution of (5)-(7). For all $\epsilon > 0$ there is a T such that $\tau \geq T$ implies $\Sigma_+(\tau) \leq 1/2 + \epsilon$.*

Proof. Let $\epsilon > 0$. If $\Sigma_+(\tau) \geq 1/2 + \epsilon/2$ for all $\tau \geq T$, then $-N_1(N_2 + N_3) \rightarrow \infty$ by (5)-(7) and Lemma 11. Since this cannot occur by (7) we conclude the existence of a $t \geq T$ such that $\Sigma_+(t) \leq 1/2 + \epsilon/2$. If there is a $s \geq t$ such that $\Sigma_+(s) \geq 1/2 + \epsilon$ there is an interval $[\tau_1, \tau_2]$ with $\tau_1 \geq t$, $\Sigma_+(\tau_1) = 1/2 + \epsilon/2$, $\Sigma_+(\tau_2) = 1/2 + \epsilon$ and $1/2 + \epsilon/2 \leq \Sigma_+(\tau) \leq 1/2 + \epsilon$ for all $\tau \in [\tau_1, \tau_2]$. Using the constraint (7) to eliminate $N_1(N_2 + N_3)$ in the expression for Σ'_+ , we conclude that

$$\frac{\epsilon}{2} = \int_{\tau_1}^{\tau_2} \Sigma'_+(\tau) d\tau \leq \frac{9}{4} \int_{\tau_1}^{\tau_2} N_1^2(\tau) d\tau.$$

Since $N'_1 \geq -2\epsilon N_1$ in $[\tau_1, \tau_2]$ we conclude that

$$\frac{\epsilon}{2} \leq \frac{9N_1^2(\tau_1)}{16\epsilon}.$$

For T great enough this inequality is impossible by Lemma 11. The lemma follows. \square

Lemma 16. *Consider a Bianchi VIII solution of (5)-(7). $N_1(N_2 + N_3)$ does not converge to zero.*

Proof. Assume $N_1(N_2 + N_3) \rightarrow 0$. We prove that this assumption forces Σ_+ to become negative which, in turn, forces $-N_1(N_2 + N_3)$ to increase, leading to a contradiction. We use Lemma 14 to achieve the decrease in Σ_+ . To apply it we need r to be big. However, the constraint yields

$$(73) \quad r^2 = 1 - \Sigma_+^2 + \frac{3}{2}N_1(N_2 + N_3) - \frac{3}{4}N_1^2.$$

Since the last two terms converge to zero due to our assumption and Lemma 11, only $1 - \Sigma_+^2$ is of relevance. From above we have control over Σ_+ by Lemma 15 and if $\Sigma_+ \leq -\frac{1}{2}$ we have nothing to prove as it turns out.

Let us be more precise. Let T be large enough that $\Sigma_+(\tau) \leq \frac{3}{4}$ for all $\tau \geq T$. This is possible by Lemma 15. Assume also T to be great enough that if $t \geq T$ and $-\frac{3}{4} \leq \Sigma_+(t)$, then

$$(74) \quad r^2(t) \geq \frac{1}{4} \geq \frac{1}{(N_2 + N_3)(t)}.$$

In order to achieve this we use (73), Lemma 11 and $N_1(N_2 + N_3) \rightarrow 0$. Finally let T be great enough that Lemma 13 and Lemma 14 are applicable to all $\tau_0 \geq T$ such that $r(\tau_0) \geq (N_2 + N_3)^{-1/2}(\tau_0)$.

Let $t \geq T$. We prove that we can iterate Σ_+ to become less than or equal to $-\frac{1}{2}$ using Lemma 14. If $\Sigma_+(t) \leq -\frac{1}{2}$ we are done. The first zero after t , say t_1 , which exists by (74) and Lemma 13, can for the same reason be assumed to satisfy $\Sigma_+(t_1) \geq -\frac{1}{2}$. Because of (65), (67), (71) and the fact that $N_1(N_2 + N_3) \rightarrow 0$, we conclude

$$(75) \quad \Sigma_+(t_2) - \Sigma_+(t_1) \leq -\frac{1}{9}(t_2 - t_1)$$

if T is great enough and t_2 is the first zero after t_1 . If $\Sigma_+(t_k) \geq -\frac{1}{2}$, the next zero t_{k+1} will satisfy a relation similar to (75). Sooner or later we will thus have $\Sigma_+(t_k) \leq -\frac{1}{2}$. Since the distance between two zeros becomes arbitrarily small as time goes on and the derivative of Σ_+ is uniformly bounded, Σ_+ cannot at late enough times become greater than $-\frac{1}{4}$ after having been smaller than $-\frac{1}{2}$ due to (75). Consequently there is a T' such that $\tau \geq T'$ implies $\Sigma_+(\tau) \leq -\frac{1}{4}$. We conclude that $-N_1(N_2 + N_3) \rightarrow \infty$ by considering the derivative of $N_1(N_2 + N_3)$ and using Lemma 11. \square

Lemma 17. *Consider a Bianchi VIII solution to (5)-(7). For all $\epsilon > 0$ there is a T such that $\tau \geq T$ implies $-N_1(N_2 + N_3) \geq 1/2 - \epsilon$.*

Proof. Assuming $-N_1(N_2 + N_3) \leq 1/2 - \epsilon/2$, we prove that the expression grows at late enough times. Under certain conditions on Σ_+ this will be seen by considering the derivative of $-N_1(N_2 + N_3)$, but under other other circumstances we apply Lemma 14. In order for the iteration to work, we will have to have a good upper bound on Σ_+ , c.f. (76). We will also need Lemma 16 in order to get a starting point greater than or equal to some fixed positive number, irrespective of at how late a time we start.

Let $\frac{1}{10} > \epsilon > 0$ and S be such that

$$(76) \quad \Sigma_+(\tau) \leq \frac{1}{2} + \epsilon^2$$

for all $\tau \geq S$. Define $w = -N_1(N_2 + N_3)$. Since w does not converge to zero there is an $\eta > 0$ such that for all S' there is a $t \geq S'$ satisfying $w(t) \geq \eta$. Let $t \geq S$ be such. We prove that there is a $t' \geq t$ with the property that

$$(77) \quad w(t') \geq \frac{1}{2} - \frac{\epsilon}{2}.$$

Assume

$$(78) \quad w(t) \leq \frac{1}{2} - \frac{\epsilon}{2}.$$

There are two possibilities. Either $\Sigma_+ \leq -\frac{1}{4}$, in which case we will be able to see that w increases by considering its derivative. If $\Sigma_+ \geq -\frac{1}{2}$ we will be able to apply Lemma 14.

1. *There is a T' such that if $[t_1, t_2]$ is an interval in which $\Sigma_+ \leq -\frac{1}{4}$ and $t_1 \geq T'$, then*

$$(79) \quad w(t_2) - w(t_1) \geq \frac{1}{2}w(t_1)(t_2 - t_1).$$

This can be seen by applying Lemma 11, (5) and (7).

2. *There is a T'' such that if $t_1 < t_2$ are two consecutive zeros of Σ_- with $T'' \leq t_1$, $\Sigma_+(t_1) \geq -\frac{1}{2}$ and $1/2 - \epsilon/2 \geq w(t_1) \geq \eta/2$, then*

$$(80) \quad w(t_2) - w(t_1) \geq \epsilon w(t_1)(t_2 - t_1).$$

We get, in t_1

$$(81) \quad r^2 = 1 - \Sigma_+^2 - \frac{3}{4}N_1^2 + \frac{3}{2}N_1(N_2 + N_3) \geq \frac{3}{4}\epsilon - 3\epsilon^2$$

if T'' is great enough. If T'' is great enough we can apply Lemma 14. Consider (66) with τ_a replaced with t_1 and τ_b replaced with t_2 . Since $2z_1^2 - 2z_1 + 2 \geq 3/2 \forall z_1 \in \mathbb{R}$ and $1/2 - \epsilon/2 \geq w(t_1)$ we get

$$w(t_2) - w(t_1) \geq \frac{3\epsilon}{2}w(t_1)(t_2 - t_1) + \epsilon_2.$$

Since $w(t_1) \geq \eta/2$ we can use (67) and (71) to prove that (80) holds if T'' is big enough. This is where we need Lemma 16.

Assume t to be greater than T' and T'' and that $w(\tau) \leq 1/2 - \epsilon/2$ for $\tau \in [t, s]$. We can divide $[t, s]$ into $t = s_0 < s_1 < s_2 < \dots < s_k < s_{k+1} = s$ where $\Sigma_+(s_i) = -\frac{1}{3}$, $i = 1, \dots, k$ and Σ_+ is either $\geq -\frac{1}{2}$ or $\leq -\frac{1}{4}$ in each $[s_i, s_{i+1}]$. If $1 \leq i \leq k$ then we can change s_i so that it becomes a zero of Σ_- without changing the earlier conditions of the construction except that the new $\Sigma_+(s_i)$ need not necessarily equal $-\frac{1}{3}$, c.f. (81). This is due to the fact that we can apply Lemma 13 if t is great enough, that $|\Sigma'_+|$ is uniformly bounded and the fact that the distance to the next zero goes to zero as t goes to infinity.

Since $w(t) \leq 1/2 - \epsilon/2$ we can apply Lemma 13 to get a zero $s_{1/2}$ with $t \leq s_{1/2}$ assuming t is great enough and $\Sigma_+(t) \geq -\frac{1}{2}$, c.f. (81). Since $s_{1/2} - t$ is of the order of magnitude $1/(N_2 + N_3)$ and w' is uniformly bounded we can assume $w(s_{1/2}) \geq \eta/2$. If $\Sigma_+ \leq -\frac{1}{4}$ in $[s_0, s_1]$, we let $s_{1/2} = t$. Observe that the condition $w(v) \geq \eta/2$ will be satisfied for a zero v of Σ_- with $s \geq v \geq s_{1/2}$ by 1 and 2. If $\Sigma_+ \geq -\frac{1}{2}$ in $[s_k, s]$ we have to take into account the possibility that s need not be a zero of Σ_- . Let

in that case $s_{k+1/2}$ be the last zero of Σ_- before s . If $\Sigma_+ \leq -\frac{1}{4}$ in $[s_k, s]$ we let $s_{k+1/2} = s$. We have, applying (80) or (79) in the intervals $[s_i, s_{i+1}]$

$$w(s_{k+1/2}) - w(s_{1/2}) \geq \frac{1}{2}\epsilon\eta(s_{k+1/2} - s_{1/2}).$$

Using the fact that $|w'|$ is uniformly bounded and the estimates of $s_{1/2} - t$ and $s - s_{k+1/2}$ we conclude that

$$w(s) - w(t) \geq \frac{1}{2}\epsilon\eta(s - t) - \frac{C}{(N_2 + N_3)(t)} - \frac{C}{(N_2 + N_3)(s)}$$

for some constant $C > 0$. We conclude the existence of a $t' \geq t$ as in (77). Finally, w cannot become smaller than $1/2 - \epsilon$ once it has become greater than $1/2 - \epsilon/2$ given t great enough. We only have to apply the above and observe that $|w'|$ is uniformly bounded and the distance between two zeros of Σ_- becomes arbitrarily small as $t \rightarrow \infty$. \square

Theorem 1. *Let $(\Sigma_+, \Sigma_-, N_1, N_2, N_3)$ be a Bianchi VIII solution to (5)-(7) with $N_1 < 0$ and $N_2, N_3 > 0$. Then*

$$\lim_{\tau \rightarrow \infty} N_1 = 0, \quad \lim_{\tau \rightarrow \infty} N_2 = \infty, \quad \lim_{\tau \rightarrow \infty} N_3 = \infty.$$

Furthermore

$$\lim_{\tau \rightarrow \infty} \Sigma_+ = \frac{1}{2}, \quad \lim_{\tau \rightarrow \infty} \Sigma_- = 0$$

and

$$\lim_{\tau \rightarrow \infty} N_1(N_2 + N_3) = -\frac{1}{2}, \quad \lim_{\tau \rightarrow \infty} (N_2 - N_3) = 0.$$

Proof. We begin by proving that for every $\epsilon > 0$ there is a T such that $\tau \geq T$ implies $\Sigma_+(\tau) \geq 1/2 - \epsilon$. We will then be able to draw the conclusions of the theorem. Use the constraint (7) to obtain

$$(82) \quad \Sigma'_+ = -\frac{3}{2}(N_2 - N_3)^2(\Sigma_+ + 1) + \frac{3}{2}N_1(N_2 + N_3)(2\Sigma_+ - 1) + \frac{3}{2}N_1^2(2 - \Sigma_+).$$

Observe that by Lemma 17 we can assume $-N_1(N_2 + N_3)$ to be big. By Lemma 11 we can disregard the term involving N_1 . If Σ_+ is smaller than, say, $1/2 - \epsilon/2$, we want Σ_+ to increase. Considering (82) two things can happen. Either r is small, in which case the first term is negligible and then the second term which is positive will dominate. Thus Σ_+ increases. If r is big, we have to use Lemma 14. We make these observations more precise in the following two statements.

1. *There is a T' such that if $r^2(\tau) \leq \epsilon/36$ and $\Sigma_+(\tau) \leq 1/2 - \epsilon/2$ for $\tau \in [t, s]$ where $t \geq T'$, then*

$$(83) \quad \Sigma_+(s) - \Sigma_+(t) \geq \frac{\epsilon}{3}(s - t).$$

The proof is as follows. If T' is great enough then $-[N_1(N_2 + N_3)](\tau) \geq \frac{1}{3}$ for $\tau \geq T'$ so that if $\Sigma_+(\tau) \leq 1/2 - \epsilon/2$ then

$$\frac{3}{2}N_1(N_2 + N_3)(2\Sigma_+ - 1) \geq \frac{\epsilon}{2}$$

in τ . Assume T' to be great enough that we also have

$$\frac{3}{2}[N_1^2(2 - \Sigma_+)](\tau) \leq \frac{\epsilon}{12}$$

for $\tau \geq T'$. If $\tau \in [t, s]$ we thus conclude that

$$\Sigma'_+(\tau) \geq \frac{\epsilon}{3}$$

using (82). The statement follows.

2. *There is a T'' such that if $r^2(\tau) \geq \epsilon/100$ and $\Sigma_+(\tau) \leq 1/2 - \epsilon/2$ for $\tau \in [\tau_1, \tau_2]$ where $\tau_1 \geq T''$ and τ_1 and τ_2 are consecutive zeros of Σ_- then*

$$(84) \quad \Sigma_+(\tau_2) - \Sigma_+(\tau_1) \geq \frac{\epsilon^2}{10}(\tau_2 - \tau_1)$$

The proof is as follows. We have

$$r^2(\tau) \geq \frac{1}{(N_2 + N_3)(\tau)}$$

for $\tau \in [\tau_1, \tau_2]$ if T'' is large enough. If T'' is large enough, we can apply Lemma 14. Consider (65). The relevant polynomial expression is

$$\begin{aligned} -(1 - z_1)(1 + z_1)^2 + \frac{3}{2}w_1(2 - z_1) &= -(1 - z_1)(1 + z_1)^2 + \frac{3}{4}(2 - z_1) \\ &+ \frac{3}{2}(w_1 - \frac{1}{2})(2 - z_1) = (z_1 + 2)(z_1 - \frac{1}{2})^2 + \frac{3}{2}(w_1 - \frac{1}{2})(2 - z_1). \end{aligned}$$

Since $-1 \leq z_1 \leq 1/2 - \epsilon/2$ in our situation, and since we can assume T'' to be great enough that $w_1 \geq 1/2 - \epsilon^2/36$ by Lemma 17, we conclude

$$-(1 - z_1)(1 + z_1)^2 + \frac{3}{2}w_1(2 - z_1) \geq \frac{\epsilon^2}{4} - \frac{\epsilon^2}{8} = \frac{\epsilon^2}{8}.$$

Thus (84) follows, in which we have absorbed the error term in (65) using (67) and (71).

Let S be greater than T' and T'' . We start by proving the existence of a $t \geq S$ such that $\Sigma_+(t) \geq 1/2 - \epsilon/2$. Assume $\Sigma_+(\tau) \leq 1/2 - \epsilon/2$ in $[S, s]$. We can divide the interval by $S = s_0 < s_1 < \dots < s_k < s_n = s$ where r^2 is either $\geq \epsilon/100$ or $\leq \epsilon/36$ in each $[s_i, s_{i+1}]$ and $r^2(s_i) = \epsilon/50$ for all s_i $i = 1, \dots, k$. If S is large enough we can assume the s_i $i = 1, \dots, k$ to be zeros of Σ_- without changing the earlier conditions of the construction other than $r^2(s_i) = \epsilon/50$, $i = 1, \dots, k$. The reason is that we can apply Lemma 13 to s_i , $i = 1, \dots, k$, if S is large enough, to get a zero within a distance of the order of magnitude $1/(N_2 + N_3)$. Using the estimate (57) and the fact that $r^2 = \tilde{x}^2 + \tilde{y}^2$, r^2 can be made to vary an arbitrarily small amount from the original s_i to the first zero after it by choosing S large enough. If $r^2(\tau) \geq \epsilon/100$ in $[s_0, s_1]$, we let $s_{1/2}$ be the first zero after s_0 . Otherwise, we let $s_{1/2} = s$. We define $s_{k+1/2}$ analogously. Using (83) and (84) we conclude

$$\Sigma_+(s_{k+1/2}) - \Sigma_+(s_{1/2}) \geq \frac{\epsilon^2}{10}(s_{k+1/2} - s_{1/2})$$

assuming $\epsilon < 10/3$. If we use our estimates of $s_{1/2} - S$ and $s - s_{k+1/2}$ we get

$$\Sigma_+(s) - \Sigma_+(S) \geq \frac{\epsilon^2}{10}(s - S) - \frac{C}{(N_2 + N_3)(S)} - \frac{C}{(N_2 + N_3)(s)}$$

If S and $s - S$ are large enough this is not possible and there must thus be a t such that $\Sigma_+(t) \geq 1/2 - \epsilon/2$.

If S big is enough, Σ_+ cannot become smaller than $1/2 - \epsilon$ once it has been larger than $1/2 - \epsilon/2$ due to (84), (83), the fact that $|\Sigma'_+|$ is uniformly bounded and the fact that the distance between two zeros of Σ_- goes to zero as $\tau \rightarrow \infty$.

Combining this observation with Lemma 15 we conclude that

$$\lim_{\tau \rightarrow \infty} \Sigma_+(\tau) = \frac{1}{2}.$$

The constraint (7) then yields

$$-N_1(N_2 + N_3) \leq \frac{2}{3}(1 - \Sigma_+^2 - \Sigma_-^2) - \frac{3}{4}(N_2 - N_3)^2 - \frac{3}{4}N_1^2 \leq \frac{2}{3}(1 - \Sigma_+^2).$$

Since the right hand side converges to $\frac{1}{2}$ we can use Lemma 17 to conclude that

$$\lim_{\tau \rightarrow \infty} N_1(N_2 + N_3) = -\frac{1}{2}.$$

By the constraint we can then conclude that

$$\lim_{\tau \rightarrow \infty} (N_2 - N_3) = 0$$

and

$$\lim_{\tau \rightarrow \infty} \Sigma_- = 0.$$

□

9. CONCLUSIONS

For all class A spacetimes except IX, q converges to a non-zero value as $\tau \rightarrow \infty$. Thus, by (27), the following theorem follows.

Theorem 2. *For all Bianchi class A vacuum spacetimes except those of type IX, the asymptotic behaviour of the reduced Hamiltonian in the expanding direction is given by*

$$\lim_{\tau \rightarrow \infty} H_{\text{reduced}}(\tau) = 0.$$

Consider the metric \tilde{g} . By (22) we have

$$\tilde{g}(\tau) = \sum_{i=1}^3 \lambda_i(\tau) \xi^i \otimes \xi^i$$

where

$$\lambda_i(\tau) = \exp\left(\int_0^\tau 6\Sigma_i d\tau'\right)$$

by (24). Note that Σ_i is a vector determined by Σ_+ and Σ_- and that the sum of its components is zero. Let us introduce some terminology.

Definition 4. If two λ_i converge to zero and one to infinity, we say that the evolution exhibits a *cigar degeneracy*. If one λ_i converges to zero and two to infinity, we call it a *pancake degeneracy*.

9.1. **Bianchi I.** In this case the Σ_i are constants satisfying

$$\sum_{i=1}^3 \Sigma_i^2 = \frac{2}{3}.$$

Thus λ_i converges to 0, ∞ or 1 depending on the solution. We get both pancakes and cigars. Concerning the a_i in (25), we observe that if the Bianchi I solution corresponds to a special point on the Kasner circle, then two a_i are constant and one goes to zero. For all other Bianchi I solutions, one a_i goes to infinity and the other two to zero.

9.2. **Bianchi II.** The Σ_i converge to s_i where the s_i satisfy

$$\sum_{i=1}^3 s_i^2 = \frac{2}{3}.$$

The points $(s_1, s_2, s_3) = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and the ones obtained by permutating its coordinates are however not allowed as limit points (they correspond to the flat Kasner vacuum solutions). Again we get pancakes or cigars depending on the solution. As far as the a_i 's are concerned, one goes to infinity and two go to zero.

9.3. **Bianchi VI₀.** In this case the Σ_i converge to $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ or one of the points obtained by permuting its coordinates. We get a cigar degeneracy.

Considering the a_i , (25) shows that one of the a_i goes to zero, but what happens to the other two is unclear.

Proposition 8. *Consider a Bianchi VI₀ solution to (5)-(7). Then all the a_i converge to zero as $\tau \rightarrow \infty$.*

Proof. Considering (25) and (26), we are interested in the integrals

$$\int_0^\tau (1 + \Sigma_+ \pm \sqrt{3}\Sigma_-)ds = \frac{1}{2} \int_0^\tau (2 + 2\Sigma_+ \pm 2\sqrt{3}\Sigma_-)ds.$$

Observe however that

$$(1 + \Sigma_+)' = -(2 - q)(1 + \Sigma_+)$$

whence

$$1 + \Sigma_+(\tau) = \exp\left[-\int_0^\tau (2 - q)ds\right][1 + \Sigma_+(0)],$$

so that, using the expressions for N_2' and N_3' in (5), we get

$$a_2^{-2} = c_2 \frac{N_2}{1 + \Sigma_+}$$

and

$$a_3^{-2} = c_3 \frac{-N_3}{1 + \Sigma_+}$$

where c_2 and c_3 are positive constants. Consider now the function

$$Z_1 = \frac{\frac{4}{3}\Sigma_-^2 + (N_2 - N_3)^2}{-N_2N_3}.$$

This function is monotone decreasing to the future, and bounded from below by 2, since

$$(N_2 - N_3)^2 \geq -2N_2N_3.$$

Thus it converges to a positive real number, and in consequence, the same is true of $1/Z_1$. Thus, using the constraint,

$$\frac{-N_2N_3}{1 + \Sigma_+} = \frac{4}{3}(1 - \Sigma_+) \frac{-N_2N_3}{\frac{4}{3}(1 - \Sigma_+^2)} = \frac{4}{3}(1 - \Sigma_+) \frac{1}{Z_1} \rightarrow \alpha$$

where α is a positive real number. Since N_2 and N_3 converge to zero as $\tau \rightarrow \infty$ by Proposition 2, we conclude that a_2 and a_3 converge to zero. \square

9.4. Bianchi VII₀. There are two possibilities. Either the Σ_i are constants equal to $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, to $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ or one of the vectors obtained by permuting the coordinates or they converge to $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ (or a permuted vector). The first possibility yields a pancake or cigar degeneracy. The second possibility yields a cigar degeneracy.

Considering the a_i we have the following proposition.

Proposition 9. *Consider a Bianchi VII₀ solution to (5)-(7). Then all the a_i converge to zero as $\tau \rightarrow \infty$.*

Proof. The argument is similar to the Bianchi VI₀ case, but easier. We just need to observe that N_2, N_3 are bounded away from zero, and $1 + \Sigma_+ \rightarrow 0$ by Lemma 6 and Proposition 5. \square

9.5. Bianchi VIII. In this case the Σ_i converge to $(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6})$ up to permutations. We have a pancake singularity.

Let us also consider what happens to the a_i appearing in (19). Considering (25), two of the a_i converge to zero, but what happens to the third one is more difficult to say, since the integrand tends to zero. Here we wish to prove that it is bounded away from infinity.

Proposition 10. *Consider a Bianchi VIII solution to (5)-(7). The a_i in (19) are bounded as $\tau \rightarrow \infty$.*

Proof. As already mentioned two of the a_i converge to zero. Let us consider the third one. Since $\Sigma_1 = -2\Sigma_+/3$, we have

$$(85) \quad a_1(\tau) = \exp\left(\int_0^\tau (2\Sigma_+ - 1)ds\right),$$

and in consequence our goal is to prove that Σ_+ does not get that much bigger than $\frac{1}{2}$. By Theorem 1, we conclude that $q - 4\Sigma_+ \rightarrow -\frac{3}{2}$, so that there is a T_1 such that

$$N_1(\tau) \leq e^{-\tau}$$

for all $\tau \geq T_1$. Using the constraint (7) to eliminate $N_1(N_2 + N_3)$, we deduce

$$\Sigma'_+ = -(1 - \Sigma_+^2 - \Sigma_-^2)(2\Sigma_+ - 1) - \frac{9}{4}(N_2 - N_3)^2 + \frac{9}{4}N_1^2.$$

Observe that in the first term, $1 - \Sigma_+^2 - \Sigma_-^2 \rightarrow \frac{3}{4}$. The term involving N_1^2 is undesirable, but it can be handled in the following way. Let

$$f = \Sigma_+ - \frac{1}{2} + N_1.$$

Then

$$f' = -(1 - \Sigma_+^2 - \Sigma_-^2)(2\Sigma_+ - 1) - \frac{9}{4}(N_2 - N_3)^2 + \frac{9}{4}N_1^2 + (q - 4\Sigma_+)N_1.$$

If $\Sigma_+(\tau) > \frac{1}{2}$, and if τ is greater than some T_2 , then we can absorb the term involving N_1^2 in the term arising from the derivative of N_1 , and conclude that

$$(86) \quad f'(\tau) \leq -\alpha f(\tau),$$

where $1 > \alpha > 0$ is a suitable constant. If there is a T such that $\Sigma_+(\tau) \leq \frac{1}{2}$ for all $\tau \geq T$, we are done, so assume not. Let us divide the problem into two subcases. Either there is a $T \geq T_1, T_2$ such that $\Sigma_+(\tau) > \frac{1}{2}$ for all $\tau \geq T$, or there is a time sequence $\tau_k \rightarrow \infty$ such that $\Sigma_+(\tau_k) = \frac{1}{2}$. It is convenient to assume $\tau_1 \geq T_1, T_2$ in this case. In the first case, we have

$$f(\tau) \leq f(T) \exp[-\alpha(\tau - T)]$$

by integrating (86), and in particular

$$0 \leq \Sigma_+(\tau) - \frac{1}{2} \leq f(T) \exp[-\alpha(\tau - T)]$$

so that the integral appearing in (85) is finite. Consider now the second subcase. Assume $\Sigma_+(\tau) > 1/2$ with $\tau > \tau_1$. Let $t < \tau$ be the first point before τ at which $\Sigma_+(t) = \frac{1}{2}$. Then

$$f(\tau) \leq f(t) \exp[-\alpha(\tau - t)] \leq \exp[-t - \alpha(\tau - t)] \leq \exp[-\alpha\tau]$$

since $\alpha < 1$ and $f(t) = N_1(t) \leq e^{-t}$. Thus the integral appearing in (85) cannot diverge to infinity. \square

ACKNOWLEDGEMENTS

This research was supported in part by the National Science Foundation under Grant No. PHY94-07194. Part of this work was carried out while the author was enjoying the hospitality of the Institute for Theoretical Physics, Santa Barbara. The author also wishes to acknowledge the support of Royal Swedish Academy of Sciences. The main motivation for writing this article originated in the work of Fischer and Moncrief, and the author is grateful to Lars Andersson for suggesting the problem and Alan Rendall for suggesting improvements.

REFERENCES

- [1] Ellis G and MacCallum M 1969 A class of homogenous cosmological models *Commun. Math. Phys.* **12** 108–41
- [2] Wainwright J and Hsu L 1989 A dynamical systems approach to Bianchi cosmologies: orthogonal models of class A *Class. Quantum Grav.* **6** 1409–31
- [3] Wainwright J, Hancock M J and Uggla C 1999 Asymptotic self-similarity breaking at late times in cosmology *Class. Quantum Grav.* **16** 2577–98
- [4] Ringström H 2000 The Bianchi IX attractor *Preprint* gr-qc/0006035
- [5] Fischer A and Moncrief V 1999 The Einstein flow, the σ -constant and the geometrization of 3-manifolds *Class. Quantum Grav.* **16** L79–L87
- [6] Barrow J and Gaspar Y 2001 Bianchi VIII empty futures *Preprint* gr-qc/0102004

- [7] Irwin M C 1980 *Smooth dynamical systems*, Academic Press

MAX-PLANCK-INSTITUT FÜR GRAVITATIONSPHYSIK, AM MÜHLENBERG 1, D-14476 GOLM, GERMANY